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SOLVING THE 4NLS WITH WHITE NOISE INITIAL DATA

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ABSTRACT. We construct global-in-time singular dynamics for the (renormalized) cubic fourth order nonlinear Schrödinger equation on the circle, having the white noise measure as an invariant measure. For this purpose, we introduce the "random-resonant / nonlinear decomposition", which allows us to single out the singular component of the solution. Unlike the classical McKean, Bourgain, Da Prato-Debussche type argument, this singular component is nonlinear, consisting of arbitrarily high powers of the random initial data. We also employ a random gauge transform, leading to random Fourier restriction norm spaces. For this problem, a contraction argument does not work and we instead establish convergence of smooth approximating solutions by studying the partially iterated Duhamel formulation under the random gauge transform. We reduce the crucial nonlinear estimates to boundedness properties of certain random multilinear functionals of the white noise.

CONTENTS

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1. Introduction

1.1. White noise on the circle and Hamiltonian partial differential equations.

A white noise on the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ is defined as the following infinite-dimensional random variable:[1]

$$u^\omega(x) = \sum_{n\in\mathbb{Z}} g_n(\omega) e^{inx}, \tag{1.1}$$

where $\{g_n\}_{n\in\mathbb{Z}}$ is a family of independent standard complex-valued Gaussian random variables. On the other hand, using the representation of the $L^2(\mathbb{T})$-norm in terms of the Fourier coefficients, one may formally define the white noise measure induced by (1.1) as

$$\text{“}Z^{-1} e^{-\frac{1}{2}\|u\|^2_{L^2(\mathbb{T})}} du\text{”}.$$

There are many important Hamiltonian PDEs such as the Korteweg-de Vries equation (KdV) and the nonlinear Schrödinger equations (NLS), under which the L^2-norm of a solution is conserved. Therefore, for this type of equations, thanks to the general globalization argument introduced by Bourgain in [6, 7], if one can solve the equation *locally in time* with data distributed according to (1.1), then one can almost surely extend the solutions for *all times* and the white noise would be an *invariant measure* of the resulting flow.

It is easy to check that the white noise measure induced by (1.1) is supported in the space of distributions $H^s(\mathbb{T}) \setminus H^{-\frac{1}{2}}(\mathbb{T})$, $s < -\frac{1}{2}$. It is this low regularity which makes it very difficult to solve locally in time a Hamiltonian PDE with the white noise initial data defined in (1.1). It is remarkable that this severe difficulty was overcome in the context of the KdV equation; see [63, 49, 50, 51, 52]. An important property of the KdV equation heavily exploited in these works is the *absence of resonant interactions* when restricted to solutions with a fixed zero Fourier mode (which is a conserved quantity for the KdV equation). As we shall see below, in the case of NLS-type equations, one may remove a part of the resonant interactions by a gauge transform. Even after such a transformation, however, there are remaining resonant interactions. The main goal of this work is to show how, by exploiting an intricate mixture of probabilistic and deterministic analysis, one may deal with such resonant interactions in the context of the cubic fourth order nonlinear Schrödinger equation on the circle with the white noise initial data (1.1). In our construction, the main random part of the solutions will be a nonlinear object (in fact, of infinite degree), which is in sharp contrast with the simple random linear evolution

[1] By convention, we endow $\mathbb{T}$ with the normalized Lebesgue measure $(2\pi)^{-1}dx$.

appearing in the previous random data well-posedness results such as [7, 15]. This difference between our main result and [7, 15] is similar in spirit with the difference between "scattering" and "modified scattering" appearing in the analysis of dispersive PDEs posed on the Euclidean space. See Remarks 1.5 and 4.3 below.

We succeeded to make our method work only for an NLS equation with a sufficiently strong dispersion. The generalization of our result to the more standard (in particular because of its integrability) NLS with the second order dispersion remains as a challenging open problem.

1.2. **The cubic fourth order nonlinear Schrödinger equation and a soft formulation of the main result.** In this work, we consider the cubic fourth order nonlinear Schrödinger equation (4NLS) on the circle $\mathbb{T}$:

$$\begin{cases} i\partial_t u = \partial_x^4 u + |u|^2 u \\ u|_{t=0} = u_0, \end{cases} \qquad (x,t) \in \mathbb{T} \times \mathbb{R}, \tag{1.2}$$

where u is complex-valued. The equation (1.2) is also called the biharmonic NLS and it was studied for instance in [38, 70] in the context of stability of solitons in magnetic materials. The L^2-norm is formally conserved by the dynamics of (1.2) and therefore, as discussed in the previous subsection, one may hope to construct global dynamics of (1.2) with data given by (1.1). This is a delicate problem for many reasons, the most basic one being that it is not clear how to interpret the nonlinearity for such low-regularity solutions.

Let us now briefly go over the deterministic well-posedness theory of (1.2). A simple fixed point argument via the Fourier restriction norm method introduced by Bourgain [5] yields local well-posedness of (1.2) in $H^s(\mathbb{T})$, $s \geq 0$. The main ingredient is the following L^4-Strichartz estimate:

$$\|u\|_{L^4(\mathbb{T}\times\mathbb{R})} \lesssim \|u\|_{X^{0,\frac{5}{16}}}, \tag{1.3}$$

where $X^{s,b}$ denotes the Fourier restriction norm space adapted to (1.2). See [57] for the proof of (1.3). Thanks to the L^2-conservation law, this local result immediately implies global well-posedness of (1.2) in $H^s(\mathbb{T})$, $s \geq 0$. The equation (1.2) is known to be ill-posed in negative Sobolev spaces in the sense of non-existence of solutions [33, 59]. See also [58, 18] for ill-posedness by norm inflation. We point out that the ill-posedness results in [58, 18] also apply to the renormalized equation (1.6) below.

Taking into account that we have a well-defined flow of (1.2) for smooth initial data, one may formulate the problem of solving (1.2) with the white noise initial data (1.1) as that of studying the limiting behavior of smooth solutions to (1.2) with initial data given by suitable regularizations of (1.1). We do not know the answer to this question in full generality but we can answer it in a satisfactory manner for the natural regularizations by mollification.

Let $\{u_{0,m}^\omega\}_{m=1}^\infty$ be a sequence of random smooth functions defined as the regularization of u^ω in (1.1) by mollification, i.e.

$$u_{0,m}^\omega = u^\omega * \rho_m = \sum_{n\in\mathbb{Z}} \widehat{\rho_m}(n) g_n(\omega) e^{inx}, \tag{1.4}$$

where $\widehat{\rho_m}(n) = \theta(n/m)$ with a bump function θ on $\mathbb{R}$ which equals one near the origin.[2] Denote by u_m the smooth solution to (1.2) with smooth initial data $u_m|_{t=0} = u_{0,m}^\omega$ constructed in

[2]We also allow θ to be a sharp cutoff function $\mathbf{1}_{[-1,1]}(n)$, in which case the resulting $u_{0,m}^\omega$ corresponds to the frequency truncated version of the white noise onto the frequencies $\{|n| \leq m\}$.

[57]. If we could solve the equation (1.2) with data given by (1.1), then the sequence $\{u_m\}_{m=1}^\infty$ would converge to the solution in an appropriate sense. The ill-posedness result in [33, 59], however, implies that there is no hope to make $\{u_m\}_{m=1}^\infty$ converge in any Sobolev space of negative regularity. It turns out that a "renormalization" of u_m is convergent. Here is a precise statement.

Theorem 1. *The sequence* $\Big\{\exp\big(2it\|u_m(t)\|_{L^2}^2\big)\,u_m(t)\Big\}_{m=1}^\infty$ *converges almost surely in*[3] $C(\mathbb{R}; H^s(\mathbb{T}))$, $s<-\frac{1}{2}$. *If we denote the limit by* u, *then we have*

$$u=\sum_{n\in\mathbb{Z}} g_n(t,\omega)e^{inx},$$

where for every $t\in\mathbb{R}$, $\{g_n(t,\omega)\}_{n\in\mathbb{Z}}$ *is a family of independent standard complex-valued Gaussian random variables. Furthermore, the limit* u *does not depend on the choice of the bump function* θ.

Theorem 1 is a satisfactory qualitative statement. It, however, does not explain in which sense the obtained limit u satisfies a limit equation and it does not give any description of the obtained limit. This will be the purpose of the next two subsections.

Remark 1.1. It is worthwhile to note that in a similar discussion for the KdV equation, one can show convergence of the sequence of regularized solutions for *any* regularization of the white noise initial data. This is because local well-posedness analysis in [39, 50] is purely deterministic. Furthermore, renormalization is not necessary for the KdV equation. It would be of interest to investigate whether the result of Theorem 1 holds for a more general class of regularizations of the white noise than those given by mollification (1.4). See Remark 1.2 for a discussion in case of smoother random initial data.

1.3. **Renormalized equation.** We now derive the equation satisfied by the limiting distribution derived in Theorem 1. Given a global solution $u\in C(\mathbb{R};L^2(\mathbb{T}))$ to (1.2), we define the following invertible gauge transform:

$$u(t)\longmapsto \mathcal{G}(u)(t):=e^{2it\fint |u(x,t)|^2dx}u(t), \tag{1.5}$$

where $\fint f(x)dx := \frac{1}{2\pi}\int_{\mathbb{T}} f(x)dx$ denotes integration with respect to the normalized Lebesgue measure $(2\pi)^{-1}dx$ on $\mathbb{T}$. A direct computation with the mass conservation shows that the gauged function, which we still denote by u, solves the following renormalized 4NLS:

$$i\partial_t u=\partial_x^4 u+\Big(|u|^2-2\fint |u|^2dx\Big)u. \tag{1.6}$$

Note that the gauge transform $\mathcal{G}$ is invertible. In particular, we can freely convert solutions to (1.2) into solutions to (1.6) and vice versa as long as they are in $C(\mathbb{R};L^2(\mathbb{T}))$. Clearly, the definition (1.5) does not make sense outside $L^2(\mathbb{T})$ (in space) and hence the original 4NLS (1.2) and the renormalized 4NLS (1.6) are no longer equivalent outside $L^2(\mathbb{T})$. As it turns out, the renormalized equation (1.6) is the one satisfied by the limiting distribution u appearing in the statement of Theorem 1.

Just like the original 4NLS (1.2), the L^4-Strichartz estimate (1.3) along with the mass conservation yields global well-posedness of the renormalized 4NLS (1.6) in $L^2(\mathbb{T})$. The important point is that the renormalization removes a certain singular component from the cubic nonlinearity; see (1.18) and (1.19) below. This allows us to study well-posedness of the renormalized

[3]Here, we endow $C(\mathbb{R}; H^s(\mathbb{T}))$ with the compact-open topology in time.

4NLS (1.6) in negative Sobolev spaces. In recent papers [41, 59], the renormalized 4NLS (1.6) was shown to be locally well-posed in $H^s(\mathbb{T})$ for $s \geq -\frac{1}{3}$ and globally well-posed for $s > -\frac{1}{3}$. Note that the white noise in (1.1) lies almost surely in $H^s(\mathbb{T}) \setminus H^{-\frac{1}{2}}(\mathbb{T})$, $s < -\frac{1}{2}$, which is beyond the scope of the known deterministic well-posedness results in [41, 59]. For this reason, the main part of our analysis is devoted to the *probabilistic* construction of local-in-time and global-in-time solutions to (1.6) with the white noise as initial data.

Note that the renormalization of the nonlinearity in (1.6) is canonical in the Euclidean quantum field theory (see, for example, [66]).[4] This formulation first appeared in the work of Bourgain [7] for studying the invariant Gibbs measure for the defocusing cubic NLS on $\mathbb{T}^2$. See [21, 54, 33, 55] for more discussion in the context of the (usual) nonlinear Schrödinger equations. See also Remark 1.6 below.

1.4. **Statements of the well-posedness results.** In the following, we consider the Cauchy problem for the renormalized 4NLS (1.6) with Gaussian random data in a more general form than (1.1). For this purpose, we introduce a family of mean-zero Gaussian measures on periodic distributions on $\mathbb{T}$. Given $\alpha \in \mathbb{R}$, consider the Gaussian measure μ_α with formal density:

$$d\mu_\alpha = Z_\alpha^{-1} e^{-\frac{1}{2}\|u\|^2_{H^\alpha}} du = Z_\alpha^{-1} \prod_{n\in\mathbb{Z}} e^{-\frac{1}{2}\langle n\rangle^{2\alpha}|\widehat{u}_n|^2} d\widehat{u}_n. \tag{1.7}$$

We can indeed view μ_α as the induced probability measure under the map Ξ_α given by

$$\Xi_\alpha : \omega \in \Omega \longmapsto \Xi_\alpha(\omega)(x) := \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{inx} \in \mathcal{D}'(\mathbb{T}), \tag{1.8}$$

where $\langle\,\cdot\,\rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$ and $\{g_n\}_{n\in\mathbb{Z}}$ is a sequence of independent standard[5] complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$. An easy computation shows that Ξ_α in (1.8) lies in $H^s(\mathbb{T})$ for

$$s < \alpha - \frac{1}{2} \tag{1.9}$$

but not in $H^{\alpha-\frac{1}{2}}(\mathbb{T})$ almost surely. In particular, μ_α is a Gaussian measure on $H^s(\mathbb{T})$ and the triplet $(H^\alpha, H^s, \mu_\alpha)$ forms an abstract Wiener space, provided that (α, s) satisfies (1.9). For more details, see [28, 40]. When $\alpha = 0$, the random Fourier series (1.8) reduces to that in (1.1) and hence the Gaussian measure μ_0 in (1.7) corresponds to the white noise measure.

Our first step is to construct local-in-time dynamics for the renormalized 4NLS (1.6) almost surely with respect to the random initial data of the form:

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{inx} \tag{1.10}$$

with $\alpha \geq 0$. For this purpose, we first introduce the following nonlinear operator Z (of infinite degree) by setting

$$Z(f)(t) := \sum_{n\in\mathbb{Z}} e^{i(nx - n^4 t)} \sum_{k=0}^{\infty} \frac{(it)^k}{k!} |\widehat{f}(n)|^{2k} \widehat{f}(n), \tag{1.11}$$

a priori defined for smooth functions $f = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{inx}$ on $\mathbb{T}$. The following theorem addresses almost sure local well-posedness of the renormalized 4NLS (1.6) for $\alpha \geq 0$.

[4] To be precise, it is an equivalent formulation to the Wick renormalization in handling rough Gaussian initial data.

[5] By convention, we set $\mathrm{Var}(g_n) = 1$.

Theorem 2 (Almost sure local well-posedness)**.** *Let $\alpha \geq 0$. Then, the renormalized cubic 4NLS* (1.6) *on $\mathbb{T}$ is locally well-posed almost surely with respect to the Gaussian measure μ_α. More precisely, there exist $C, c > 0$ such that for each sufficiently small $\delta > 0$, there exists a set $\Omega_\delta \subset \Omega$ with the following properties:*

(i) *$P(\Omega_\delta^c) = \mu_\alpha \circ \Xi_\alpha(\Omega_\delta^c) < Ce^{-\frac{1}{\delta^c}}$, where μ_α and Ξ_α are as in* (1.7) *and* (1.8).

(ii) *For each $\omega \in \Omega_\delta$, there exists a (unique) solution u to* (1.6) *with $u|_{t=0} = u_0^\omega$ given by the random Fourier series* (1.10) *in the class:*
$$z^\omega + C([-\delta, \delta]; L^2(\mathbb{T})) \subset C([-\delta, \delta]; H^s(\mathbb{T})), \tag{1.12}$$
where $z^\omega = Z(u_0^\omega)$ is as in (1.11) *and* (i) *$s = 0$ if $\alpha > \frac{1}{2}$ and* (ii) *$s = \alpha - \frac{1}{2} - \varepsilon$ for any $\varepsilon > 0$, if $\alpha \leq \frac{1}{2}$.*

In the next subsections, we discuss an outline of the proof of Theorem 2.

Remark 1.2. (i) When $\alpha > \frac{1}{2}$, the random initial data u_0^ω in (1.10) belongs almost surely to $L^2(\mathbb{T})$ and hence the deterministic uniqueness statements apply. In particular, when $\alpha > \frac{2}{3}$, one can easily modify the argument in [32] to conclude that the solution to (1.6) is almost surely unconditionally unique, namely, uniqueness holds in the entire $C([-\delta, \delta]; H^{\frac{1}{6}}(\mathbb{T}))$. For $\frac{1}{2} < \alpha \leq \frac{2}{3}$, the solution is almost surely conditionally unique. Namely, uniqueness holds in an auxiliary function space (the $X^{0,b}$-space for some $b > \frac{1}{2}$ in this case) contained in $C([-\delta, \delta]; L^2(\mathbb{T}))$. As for the uniqueness statements for $0 \leq \alpha \leq \frac{1}{2}$, see Remark 1.10 for $0 < \alpha \leq \frac{1}{2}$ and Remark 4.4 for $\alpha = 0$.

(ii) Given $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, define the Fourier-Lebesgue space $\mathcal{F}L^{s,p}(\mathbb{T})$ by the norm:
$$\|f\|_{\mathcal{F}L^{s,p}(\mathbb{T})} := \|\langle n \rangle^s \widehat{f}(n)\|_{\ell^p_n(\mathbb{Z})}. \tag{1.13}$$
Then, given $\alpha \in \mathbb{R}$, it is easy to see that the random initial data $u_0^\omega = u_0^\omega(\alpha)$ in (1.10) belongs almost surely to the Fourier-Lebesgue space $\mathcal{F}L^{s,p}(\mathbb{T})$ defined in (1.13) if and only if $(s-\alpha)p < 1$, namely $s < \alpha - \frac{1}{p}$. See [1] for example. In particular, given $\alpha > 0$, we can take $s \geq 0$ by choosing sufficiently large $p = p(\alpha) \gg 1$. Then, as already pointed out in [21], by adapting deterministic local well-posedness results [19, 29, 60] of the renormalized cubic NLS (with the second order dispersion) in the Fourier-Lebesgue space $\mathcal{F}L^{s,p}(\mathbb{T})$ with $s \geq 0$ and $1 \leq p < \infty$ to the renormalized 4NLS (1.6), we can prove almost sure local well-posedness of (1.6) with the random initial data $u_0^\omega = u_0^\omega(\alpha)$ in (1.10) (via the deterministic method), provided that $\alpha > 0$. As in the case of KdV discussed in Remark 1.1, we then have convergence of the sequence of regularized solutions for *any* regularization of the initial data in the appropriate Fourier-Lebesgue space, when $\alpha > 0$. See also [25] for an analogous local well-posedness result in the context of the stochastic cubic NLS on $\mathbb{T}$ with almost space-time white noise.

When $\alpha = 0$, however, the white noise defined in (1.1) belongs almost surely to $\mathcal{F}L^{s,p}(\mathbb{T})$ only for $s < -\frac{1}{p}$ and thus the deterministic argument in [19, 29, 60] is no longer applicable to our problem. In fact, our main goal in this paper is to prove Theorem 2 when $\alpha = 0$, which requires a new idea. See Sections 4 and 6.

Lastly, we point out that the construction of a solution in Theorem 2 is done in the more canonical Sobolev space $H^s(\mathbb{T})$ (rather than the Fourier-Lebesgue space $\mathcal{F}L^{s,p}(\mathbb{T})$) and this presents a challenge even for $\alpha > 0$, as it was observed in the paper by Colliander and the first author [21] in the case of the standard (renormalized) cubic NLS with the random initial data of the form (1.10). See Section 3.

Theorem 2 with $\alpha = 0$ shows that the renormalized 4NLS (1.6) is almost surely locally well-posed with the white noise in (1.1) as initial data. In constructing almost sure global-in-time dynamics, we adapt Bourgain's invariant measure argument [6, 7] to our setting. More precisely, we use invariance of the white noise measure under the finite-dimensional approximation of the 4NLS flow to obtain a uniform control on the solutions, and then apply a PDE approximation argument to extend the local solutions to (1.6) obtained from Theorem 2 to global ones. As a byproduct, we also obtain invariance of the white noise under the resulting global flow of the renormalized 4NLS (1.6).

Theorem 3 (Almost sure global well-posedness and invariance of the white noise)**.** *Let $\alpha = 0$. Then, the renormalized 4NLS* (1.6) *on $\mathbb{T}$ is globally well-posed almost surely with the random initial data u_0^ω given by* (1.10)*. More precisely, for almost every $\omega \in \Omega$, there exists a unique solution u to* (1.6) *with $u|_{t=0} = u_0^\omega$, satisfying*

$$u \in z^\omega + C(\mathbb{R}; L^2(\mathbb{T})) \subset C(\mathbb{R}; H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}))$$

for any $\varepsilon > 0$, where $z^\omega = Z(u_0^\omega)$. Furthermore, the white noise measure μ_0 is invariant under the flow.

Remark 1.3. When $\alpha > \frac{1}{6}$, the deterministic global well-posedness [59] of the renormalized 4NLS (1.6) in $H^s(\mathbb{T})$, $s > -\frac{1}{3}$, implies almost sure global well-posedness of (1.6) with the random initial data u_0^ω in (1.10) since the random initial data u_0^ω almost surely belongs to $H^s(\mathbb{T})$ for some $s > -\frac{1}{3}$.

The proof of Theorem 3 heavily depends on (formal) invariance of the white noise measure and hence is not applicable for the case $\alpha \in (0, \frac{1}{6}]$. In [21], Colliander and the first author adapted Bourgain's high-low decomposition method [9] to prove almost sure global well-posedness of the renormalized NLS (with the second order dispersion) with the random initial data of the form (1.10) below $L^2(\mathbb{T})$ (without relying on any invariant measure). The same approach is expected to yield almost sure global well-posedness of the renormalized 4NLS (1.6) for some range of $\alpha \in (0, \frac{1}{6}]$. We do not pursue this analysis here.

Remark 1.4. The solution u constructed in Theorems 2 and 3 has a structure:

$$u = \text{random } \textit{nonlinear} \text{ term} + \text{smoother term}.$$

See (1.38). This is quite different from the standard probabilistic well-posedness results as in [7, 15], where a solution u has the structure:

$$u = \text{random linear term} + \text{smoother term}. \tag{1.14}$$

In the field of stochastic PDEs, a well-posedness argument based on the decomposition (1.14) is usually referred to as the Da Prato-Debussche trick. When the decomposition (1.14) is not sufficient, one may try to write a solution as the sum of *finitely many* stochastic terms plus a smoother remainder. See for example [30, 34].

In the context of nonlinear dispersive PDEs, there are recent works [3, 53], where a solution theory was built, based on the decomposition of a solution as the sum of finitely many stochastic terms plus a smoother remainder. A remarkable new feature of the decomposition used in Theorems 2 and 3 is that the series expansion (1.11) for $Z(u_0^\omega)$ consists not only of the free solution (i.e. $k = 0$ in (1.11)) but also of *infinitely many* higher order corrections terms $k \geq 1$. As a consequence, $z^\omega = Z(u_0^\omega)$ depends on arbitrarily high powers of Gaussian random variables and hence it does *not* belong to Wiener chaoses $\mathcal{H}_{\le k}$, defined in (2.10), of any finite order. See also Remark 1.11.

Remark 1.5. A decomposition such as (1.14) is not only useful in establishing well-posedness of a given equation, but also provides a finer regularity description of a solution thus obtained. For example, The decomposition (1.14) states that in the high frequency regime (i.e. at small spatial scales on the physical side), the dynamics is essentially governed by that of the random linear solution. See also Remark 1.11 (ii). In [10, Page 62], Bourgain made an "analogy" of the decomposition (1.14) to *scattering* (i.e. a nonlinear solution behaving like a linear solution asymptotically as $t \to \pm\infty$) by saying "This property [namely the decomposition (1.14)] reminds of "scattering" occurring in certain dispersive models" in the sense that in both the decomposition (1.14) and scattering, the dominant part of dynamics is given by the linear dynamics.

In our solution theory, we have the decomposition

$$u = z^\omega + \text{ smoother term},$$

where $z^\omega = Z(u_0^\omega)$. Namely, the dominant part is *nonlinear* (with an explicit structure). In this context, one may wish to say that the results of Theorems 2 and 3 remind of *modified scattering* occurring in certain dispersive models [61, 37, 36], where the asymptotic dominant dynamics is given not by a linear dynamics but by a certain nonlinear dynamics. See Remark 4.3 below for more details on this analogy.

Remark 1.6. Instead of the renormalized 4NLS (1.6), one may work with the Wick renormalization to study the same problem. Disadvantage for this approach is that there is no equation for the limiting dynamics. The limit u of smooth approximating solutions would formally "satisfy"

$$i\partial_t u = \partial_x^4 u + |u|^2 u - \infty \cdot u. \tag{1.15}$$

This is in sharp contrast with the case of the renormalized 4NLS (1.6), where the renormalized nonlinearity has a well defined meaning as a cubic operator, defined a priori on smooth functions. See (1.18) and (1.19). Lastly, we point out that if the Gaussian measure μ_α in (1.7) were invariant, then one could show that the renormalized 4NLS (1.6) is equivalent to the Wick ordered 4NLS (1.15) in a suitable limiting sense, provided that $\alpha > \frac{1}{4}$. See Section 3 in [54]. Unfortunately, such invariance is true only for $\alpha = 0$.

1.5. **Outline of the well-posedness argument.** When $\alpha > \frac{1}{2}$, it follows from (1.9) that our random initial data u_0^ω defined in (1.10) belongs to $L^2(\mathbb{T})$ almost surely. Hence, the aforementioned deterministic global well-posedness of (1.6) in $L^2(\mathbb{T})$ implies Theorem 2 in this case. Therefore, we focus on the case $0 \le \alpha \le \frac{1}{2}$ in the following.

When $0 \le \alpha \le \frac{1}{2}$, the random initial data u_0^ω in (1.10) lies strictly in negative Sobolev spaces almost surely. In view of the failure of the local uniform continuity of the solution map in these spaces (see [21, 57]), it is non-trivial to construct solutions to (1.6) in negative Sobolev spaces since a straightforward contraction argument fails in this regime. For $\alpha > \frac{1}{6}$, the random initial data u_0^ω in (1.10) almost surely belongs to $H^s(\mathbb{T})$ for some $s > -\frac{1}{3}$ and hence the global well-posedness in [59] based on a more robust energy method is applicable to conclude Theorem 2. In the following, however, we present a uniform approach to construct local-in-time solutions in a probabilistic manner for $0 \le \alpha \le \frac{1}{2}$ by making use of randomness of the initial data u_0^ω in (1.10).

By writing (1.6) in the Duhamel formulation, we have

$$u(t) = S(t)u_0^\omega - i\int_0^t S(t-t')\mathcal{N}(u)(t')dt', \tag{1.16}$$

where $S(t) = e^{-it\partial_x^4}$ denotes the linear propagator and

$$\mathcal{N}(u) = \bigg(|u|^2 - 2\fint |u|^2 dx\bigg)u. \tag{1.17}$$

Next, we make an important decomposition of the nonlinearity $\mathcal{N}(u)$ into resonant and non-resonant parts. Namely, define trilinear operators $\mathcal{N}_1$ and $\mathcal{N}_2$ by setting

$$\mathcal{N}_1(u_1,u_2,u_3)(x,t) := \sum_{n\in\mathbb{Z}}\sum_{\Gamma(n)} \widehat{u}_1(n_1,t)\overline{\widehat{u}_2(n_2,t)}\widehat{u}_3(n_3,t)e^{i(n_1-n_2+n_3)x}, \tag{1.18}$$

$$\mathcal{N}_2(u_1,u_2,u_3)(x,t) := -\sum_{n} \widehat{u}_1(n,t)\overline{\widehat{u}_2(n,t)}\widehat{u}_3(n,t)e^{inx}, \tag{1.19}$$

where $\Gamma(n)$ denotes the hyperplane:

$$\Gamma(n) := \big\{(n_1,n_2,n_3)\in\mathbb{Z}^3 : \, n = n_1 - n_2 + n_3 \text{ and } n_1, n_3 \neq n\big\}. \tag{1.20}$$

When all the arguments coincide, we simply write $\mathcal{N}_k(u) = \mathcal{N}_k(u,u,u)$, $k = 1,2$. The term $\mathcal{N}_1(u)$ denotes the non-resonant part of the renormalized nonlinearity $\mathcal{N}(u)$, while $\mathcal{N}_2(u)$ denotes the resonant part. Then, the renormalized nonlinearity $\mathcal{N}(u)$ can be written as

$$\mathcal{N}(u) = \mathcal{N}_1(u) + \mathcal{N}_2(u).$$

Let us first go over the basic idea of the probabilistic local well-posedness, as developed for instance in [7, 15, 68, 21, 45]. See also [42]. This argument is based on the following first order expansion:

$$u = z_1^\omega + v, \tag{1.21}$$

where z_1^ω denotes the random linear solution defined by

$$z_1^\omega(t) := S(t)u_0^\omega. \tag{1.22}$$

By rewriting (1.16) as a fixed point problem for the residual term $v := u - z_1^\omega$, we obtain the following perturbed renormalized 4NLS:

$$v(t) = -i\int_0^t S(t-t')\mathcal{N}(v + z_1^\omega)(t')dt'. \tag{1.23}$$

Then, the main aim is to solve this fixed point problem for v in $L^2(\mathbb{T})$,[6] where the unperturbed equation (1.6) is deterministically well-posed by a simple contraction argument. In particular, it is crucial to make use of probabilistic tools (for example, see Subsection 2.2) and show that the perturbation $\mathcal{N}(v + z_1^\omega) - \mathcal{N}(v)$ is *smoother* than the random linear solution z_1^ω and lies in $L^2(\mathbb{T})$ for each t. When $\alpha > \frac{1}{6}$, this can be indeed achieved and we can show that for each small $\delta > 0$, there exists $\Omega_\delta \subset \Omega$ with $P(\Omega_\delta^c) < Ce^{-\frac{1}{\delta^c}}$ such that for each $\omega \in \Omega_\delta$, there exists a solution $u = z_1^\omega + v$ to the renormalized 4NLS (1.6) in the class:

$$z_1^\omega + C([-\delta,\delta]; L^2(\mathbb{T})) \subset C([-\delta,\delta]; H^s(\mathbb{T})),$$

[6]Strictly speaking, we need to consider the fixed point problem (1.23) in some appropriate function space $X_\delta \subset C([-\delta,\delta]; L^2(\mathbb{T}))$. For simplicity, however, we only discuss the spatial regularity and suppress its time dependence. A similar comment applies in the following. In particular, in discussing spatial regularity of a space-time distribution, we may suppress its time dependence.

for $s < \alpha - \frac{1}{2}$. The most singular contribution on the right-hand side of (1.23) is given by

$$z_3^\omega(t) := -i \int_0^t S(t-t')\mathcal{N}_2(z_1^\omega)(t')dt' = it \sum_{n\in\mathbb{Z}} \frac{|g_n|^2 g_n}{\langle n\rangle^{3\alpha}} e^{i(nx-n^4t)} \tag{1.24}$$

where $\mathcal{N}_2$ is as in (1.19), denoting the resonant interaction. This resonant cubic[7] term is responsible for the restriction $\alpha > \frac{1}{6}$. It is easy to see that $z_3^\omega(t)$ lies in $H^s(\mathbb{T}) \setminus H^{3\alpha-\frac{1}{2}}(\mathbb{T})$ almost surely for

$$s < 3\alpha - \frac{1}{2}.$$

In particular, when $\alpha > \frac{1}{6}$, the L^2-deterministic well-posedness theory (via a contraction argument) becomes available for solving the perturbed equation (1.23). As mentioned above, the case $\alpha > \frac{1}{6}$ is also covered by the deterministic well-posedness in [41, 59] (based on a more robust energy method) and thus our main goal in the following is to treat lower values of α.

Remark 1.7. This argument is basically the Da Prato-Debussche trick in the context of stochastic PDEs [22, 23], where the random linear solution is replaced by the solution to a linear stochastic PDE. See [35] for a concise discussion on the Da Prato-Debussche trick. It is worthwhile to point out that the paper [42, 7] by McKean and Bourgain precede [22, 23].

According to the discussion above, the basic probabilistic argument based on the first order expansion (1.21) does not work for our problem when $\alpha \le \frac{1}{6}$ because the second order term z_3^ω does not belong to $L^2(\mathbb{T})$ almost surely if $\alpha \le \frac{1}{6}$. See also Case (b) in Subsection 4.2 of [21]. This shows that we can not solve the fixed point problem (1.23) in $L^2(\mathbb{T})$ when $\alpha \le \frac{1}{6}$.

A natural next step would be to consider the following second order expansion:

$$u = z_1^\omega + z_3^\omega + v$$

for a solution u to (1.6) and study the equation satisfied by the residual term $v := u - z_1^\omega - z_3^\omega$:

$$\begin{cases} i\partial_t v = \partial_x^4 v + \big[\mathcal{N}(v + z_1^\omega + z_3^\omega) - \mathcal{N}_2(z_1^\omega)\big] \\ v|_{t=0} = 0. \end{cases}$$

Namely, we consider the following fixed point problem:

$$v(t) = -i \int_0^t S(t-t')\big[\mathcal{N}(v + z_1^\omega + z_3^\omega) - \mathcal{N}_2(z_1^\omega)\big](t')dt'. \tag{1.25}$$

Note that the worst contribution z_3^ω in the first step coming from the resonant interaction $\mathcal{N}_2(z_1^\omega)$ is now eliminated. We can then perform case-by-case nonlinear analysis on $\mathcal{N}_k(u_1, u_2, u_3)$, $k = 1, 2$, in the spirit of [7, 21], where each u_j can be z_1^ω, z_3^ω, or the smoother unknown function v except for the case $u_1 = u_2 = u_3 = z_1^\omega$ with $k = 2$. This allows us to show that the fixed point problem (1.25) for the residual term v is almost surely locally well-posed in $L^2(\mathbb{T})$, provided that $\alpha > \frac{1}{10}$. Recalling that $z_1^\omega, z_3^\omega \in C(\mathbb{R}; H^s(\mathbb{T}))$ for s satisfying (1.9), we obtain a solution $u = z_1 + z_3 + v$ to the renormalized 4NLS (1.6) in the class:

$$z_1^\omega + z_3^\omega + C([-\delta,\delta]; L^2(\mathbb{T})) \subset C([-\delta,\delta]; H^s(\mathbb{T}))$$

almost surely, for $s < \alpha - \frac{1}{2}$.

[7] Namely, z_3^ω in (1.24) is trilinear in the random initial data.

In this second step, the restriction $\alpha > \frac{1}{10}$ comes from the following resonant quintic term in (1.25):

$$\begin{aligned} z_5^\omega(t) &:= -i \sum_{\substack{j_1, j_2, j_3 \in 2\mathbb{N}-1 \\ j_1+j_2+j_3 = 5}} \int_0^t S(t-t')\mathcal{N}_2(z_{j_1}^\omega, z_{j_2}^\omega, z_{j_3}^\omega)(t')dt' \\ &= -\frac{t^2}{2} \sum_{n\in\mathbb{Z}} \frac{|g_n|^4 g_n}{\langle n\rangle^{5\alpha}} e^{i(nx - n^4 t)}. \end{aligned} \tag{1.26}$$

Given $t \in \mathbb{R}$, it is easy to see that $z_5^\omega(t)$ lies in $H^s(\mathbb{T}) \setminus H^{5\alpha-\frac{1}{2}}(\mathbb{T})$ almost surely for

$$s < 5\alpha - \frac{1}{2}.$$

In particular, $z_5^\omega(t)$ does not lie in $L^2(\mathbb{T})$ almost surely if $\alpha \le \frac{1}{10}$.

One can repeat this process in an obvious manner. Namely, consider the following third order expansion:

$$u = z_1^\omega + z_3^\omega + z_5^\omega + v$$

for a solution u to (1.16) and study the fixed point problem for $v = u - z_1^\omega - z_3^\omega - z_5^\omega$. From the discussion above, we see that the limitation comes from the resonant septic term, yielding the restriction of $\alpha > \frac{1}{14}$.

In general, in the kth step, we could write a solution u to (1.16) as

$$u = v + \sum_{j=1}^k z_{2j-1}^\omega \tag{1.27}$$

and consider the fixed point problem for $v = u - \sum_{j=1}^k z_{2j-1}^\omega$. Here, z_{2j-1} denotes the following resonant $(2j-1)$-linear term (in the random initial data):

$$z_{2j-1}^\omega(t) := -i \sum_{\substack{j_1, j_2, j_3 \in 2\mathbb{N}-1 \\ j_1+j_2+j_3 = 2j-1}} \int_0^t S(t-t')\mathcal{N}_2(z_{j_1}^\omega, z_{j_2}^\omega, z_{j_3}^\omega)(t')dt'. \tag{1.28}$$

Proceeding as before, it is easy to see that the limitation in this kth step comes from z_{2k+1}^ω yielding the restriction of

$$\alpha > \frac{1}{2(2k+1)} \tag{1.29}$$

which is needed to guarantee that $z_{2k+1}^\omega(t)$ belongs almost surely to $L^2(\mathbb{T})$.

The restriction (1.29) shows that, in order to treat the $\alpha = 0$ case, we *at least* need an infinite iteration of this procedure. Furthermore, the argument based on the kth order expansion (1.27) leads to the following equation for the the residual term $v = u - \sum_{j=1}^k z_{2j-1}^\omega$:

$$\begin{cases} i\partial_t v = \partial_x^4 v + \mathcal{N}\Big(v + \sum_{j=1}^k z_{2j-1}^\omega\Big) - \sum_{\substack{j_1+j_2+j_3 \in \{3,5,\dots,2k-1\} \\ j_1,j_2,j_3 \in \{1,3,\dots,2k-3\}}} \mathcal{N}_2(z_{j_1}^\omega, z_{j_2}^\omega, z_{j_3}^\omega) \\ v|_{t=0} = 0. \end{cases}$$

In particular, we need to carry out the following case-by-case nonlinear analysis on

$$\mathcal{N}_\ell(u_1, u_2, u_3), \quad \ell = 1, 2,$$

where each u_i, $i = 1, 2, 3$, can be either the smoother unknown function v or z_j^ω for some $j \in \{1, 3, \dots, 2k-1\}$ such that it is not of the form $\mathcal{N}_2(z_{j_1}, z_{j_2}, z_{j_3})$ with $j_1+j_2+j_3 \in \{3, 5, \dots, 2k-1\}$. In general, it could be a cumbersome task to carry out this case-by-case analysis due to the increasing number of combinations. In the next subsection, we will describe an approach to overcome this issue.

Remark 1.8. In [3], the first author with Bényi and Pocovnicu studied the cubic NLS on $\mathbb{R}^3$ with random initial data based on a higher order expansion (of order k), analogous to (1.27). In order to avoid a combinatorial nightmare in relevant case-by-case analysis for high values of k, the authors introduced a modified expansion of order k, which simplified the relevant analysis in a significant manner. We point out that the analysis in [3] is significantly simpler than that in the current paper, since (i) the random data considered in [3] are of positive regularities and (ii) the refinement of the bilinear Strichartz estimates [9, 62] are available on the Euclidean space. We also mention a recent work [53] on the probabilistic local well-posedness of the three-dimensional cubic nonlinear wave equation in negative Sobolev spaces, where the main analysis is based on the second order expansion.

1.6. **The $\alpha > 0$ case.** In this subsection, we describe an outline of the proof of Theorem 2 for the $\alpha > 0$ case. In the next subsection, we discuss additional ingredients required to treat the $\alpha = 0$ case.

In view of the restriction (1.29), we need to iterate indefinitely the procedure described above in order to treat arbitrary $\alpha > 0$. For this purpose, we define z^ω by

$$z^\omega = \sum_{j=1}^{\infty} z_{2j-1}^\omega. \tag{1.30}$$

Then, from (1.22), (1.24), (1.26), and (1.28), we see that z^ω defined in (1.30) is nothing but a power series expansion of a solution to the following *resonant* 4NLS:

$$\begin{cases} i\partial_t z^\omega = \partial_x^4 z^\omega + \mathcal{N}_2(z^\omega) \\ z^\omega|_{t=0} = u_0^\omega, \end{cases} \tag{1.31}$$

where u_0^ω is the random initial data defined in (1.10). By letting $\mathsf{z}(t) = S(-t)z^\omega(t)$, we see that $\widehat{\mathsf{z}}_n(t) = \widehat{\mathsf{z}}(n, t)$ satisfies the following ODE:

$$\begin{cases} i\partial_t \widehat{\mathsf{z}}_n = -|\widehat{\mathsf{z}}_n|^2 \widehat{\mathsf{z}}_n \\ \widehat{\mathsf{z}}_n|_{t=0} = \frac{g_n}{\langle n \rangle^\alpha}, \end{cases} \tag{1.32}$$

for each $n \in \mathbb{Z}$. By the explicit formula of solutions to (1.32), we have

$$\widehat{\mathsf{z}}_n(t) = e^{it|\widehat{\mathsf{z}}_n(0)|^2} \widehat{\mathsf{z}}_n(0). \tag{1.33}$$

Hence, we can express z^ω as

$$z^\omega(t) = \sum_{n \in \mathbb{Z}} e^{i(nx - n^4 t)} e^{it\frac{|g_n|^2}{\langle n \rangle^{2\alpha}}} \frac{g_n}{\langle n \rangle^\alpha}. \tag{1.34}$$

By expanding in a power series, we obtain

$$z^\omega(t) = \sum_{n \in \mathbb{Z}} e^{i(nx - n^4 t)} \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \frac{|g_n|^{2k} g_n}{\langle n \rangle^{(2k+1)\alpha}}. \tag{1.35}$$

By comparing (1.11) and (1.35) with (1.10), we obtain

$$z^\omega = Z(u_0^\omega).$$

Note that, unlike the random linear solution z_1^ω in (1.22) and other lower order terms z_{2j-1}^ω in (1.28), the random resonant solution z^ω depends on arbitrarily high powers of Gaussian random variables and hence it does *not* belong to Wiener chaoses of any finite order. Nonetheless, the formula (1.34) shows that z^ω has a particular simple structure, allowing us to study its regularity properties; see Lemmas 1.9 and 2.10 below. In carrying out analysis on the random resonant solution z^ω involving the $X^{s,b}$-spaces, we instead need to make use of the series expansion (1.35) and apply Lemma 2.11 below for each k.

Lemma 1.9. *Given $\alpha \in \mathbb{R}$, let z^ω be as in* (1.34). *Then, z^ω belongs to $C(\mathbb{R}; H^s(\mathbb{T}))$ almost surely, provided that $s < \alpha - \frac{1}{2}$.*

Proof. Fix $\varepsilon > 0$ sufficiently small such that

$$s + \varepsilon < \alpha - \tfrac{1}{2}. \tag{1.36}$$

Lemma 2.7 below states that we have

$$\sup_{n\in\mathbb{Z}} |g_n(\omega)| \le C(\omega)\langle n\rangle^\varepsilon \tag{1.37}$$

for some almost surely finite constant $C(\omega) > 0$.

For fixed $t \in \mathbb{R}$, let $\{t_j\}_{j=1}^\infty$ be a sequence converging to t. Then, for each $n \in \mathbb{Z}$, it follows from (1.33) that $\widehat{z^\omega}(n, t_j)$ converges to $\widehat{z^\omega}(n, t)$ almost surely as $j \to \infty$. Furthermore, from (1.33) and (1.37), we have

$$\sup_{j\in\mathbb{N}} \langle n\rangle^s |\widehat{z^\omega}(n, t_j)| + \langle n\rangle^s |\widehat{z^\omega}(n, t)| \le 2C(\omega)\langle n\rangle^{s-\alpha+\varepsilon},$$

where the right-hand side belongs to $\ell^2(\mathbb{Z})$ in view of (1.36). Hence, the claim follows from the dominated convergence theorem. □

Now, express a solution u to (1.6) in the following *random-resonant / nonlinear decomposition*:

$$u = z^\omega + v. \tag{1.38}$$

Then, the residual term $v = u - z^\omega$ satisfies

$$\begin{cases} i\partial_t v = \partial_x^4 v + \big[\mathcal{N}(v + z^\omega) - \mathcal{N}_2(z^\omega)\big] \\ v|_{t=0} = 0. \end{cases} \tag{1.39}$$

By writing (1.39) in the Duhamel formulation, we consider the following fixed point problem:

$$v(t) = \Gamma^\omega v(t) := -i\int_0^t S(t - t')\big[\mathcal{N}(v + z^\omega) - \mathcal{N}_2(z^\omega)\big](t')dt'. \tag{1.40}$$

In this formulation, we successfully reduced the number of combinations; we only need to study $\mathcal{N}_k(u_1, u_2, u_3)$, $k = 1, 2$, where each u_j can be either the random resonant solution z^ω or the smoother unknown function v, except for the case $u_1 = u_2 = u_3 = z^\omega$ with $k = 2$. In Section 3, we perform the case-by-case nonlinear analysis and show that the fixed point problem (1.40) is almost surely locally well-posed in $L^2(\mathbb{T})$ via the standard Fourier restriction norm method, provided that $\alpha > 0$.

Lastly, Lemma 1.9 allows us to conclude that the solution $u = z^\omega + v$ to the renormalized 4NLS (1.6) lies in the class:

$$z^\omega + C([-\delta, \delta]; L^2(\mathbb{T})) \subset C([-\delta, \delta]; H^s(\mathbb{T}))$$

almost surely.

Remark 1.10. The probabilistic local well-posedness argument in [7, 15, 68, 21] yields uniqueness of solutions in a ball of radius $O(1)$ in a suitable (local-in-time) function space (such as the Strichartz spaces or the $X^{s,b}$-spaces) centered at the random linear solution. When $\alpha > 0$, the proof of Theorem 2 yields uniqueness of solutions in the ball of radius 1 in $X^{0,\frac{1}{2}+,\delta}$ centered at the random resonant solution z^ω.

Remark 1.11. (i) When $\alpha > 0$, the terms z^ω_{2j-1} appearing in (1.27) get smoother as j increases and hence only a finite number of expansion is needed. Nonetheless, the random-resonant / nonlinear decomposition (1.38) allows us to avoid a number of combinations in the relevant case-by-case analysis when $k \gg 1$. When $\alpha = 0$, the terms z^ω_{2j-1} in (1.30) do *not* get smoother and hence the infinite order expansion in (1.30) is necessary in this case.
(ii) Let $\alpha > 0$. In this case, the random-resonant / nonlinear decomposition (1.38) with (1.30) allows us to write the solution u as

$$u = z_1^\omega + z_3^\omega + \cdots + z_{2k+1}^\omega + v \tag{1.41}$$

for some $v \in C([-\delta, \delta]; L^2(\mathbb{T}))$, where k is the smallest non-negative integer such that (1.29) holds. The expansion (1.41) provides a finer regularity description[8] of the solution u than the random-linear / nonlinear decomposition (1.21). As mentioned above, the terms in (1.30) do not get smoother when $\alpha = 0$. In this case, the solution u can be written as

$$u = z^\omega + v$$

for some $v \in C([-\delta, \delta]; L^2(\mathbb{T}))$. Namely, the dominant part of the dynamics in small scales is indeed given by the random resonant solution z^ω defined in (1.34).

1.7. **The $\alpha = 0$ case.** Next, let us discuss the $\alpha = 0$ case. Namely, we consider the white noise initial data (1.1). Unfortunately, the argument described above breaks down in this case. As we see in Section 3, the worst interaction comes from the following *resonant nonlinear* terms on the right-hand side of (1.39):

$$\mathcal{N}_2(v, z^\omega, z^\omega) + \mathcal{N}_2(z^\omega, z^\omega, v) = -2\mathcal{F}^{-1}\big[|g_n|^2 \widehat{v}(n)\big]$$

and

$$\mathcal{N}_2(z^\omega, v, z^\omega) = -\mathcal{F}^{-1}\Big[e^{-2in^4t} e^{2it|g_n|^2} g_n^2\, \overline{\widehat{v}(n)}\Big].$$

In order to weaken the effect of these terms, we introduce the following *random* gauge transform:

$$\mathcal{J}^\omega(u)(x,t) = \sum_{n\in\mathbb{Z}} e^{inx - it|g_n(\omega)|^2} \widehat{u}(n,t). \tag{1.42}$$

When $\alpha = 0$, the solution z^ω to the resonant 4NLS (1.31) reads as

$$z^\omega(x,t) = \sum_{n\in\mathbb{Z}} e^{i(nx - n^4 t)} e^{it|g_n|^2} \widehat{u_0^\omega}(n). \tag{1.43}$$

[8]This regularity description can also be understood as the "local" (in space) description of the solution since the singular components of the solution become dominant in small scales.

The random gauge transform $\mathcal{J}^\omega$ in (1.42) allows us to filter out the random phase oscillations appearing in (1.43). This gauge transform is clearly invertible and leaves the H^s-norm invariant. If u is a solution to the renormalized 4NLS (1.6), then the gauged function $w := \mathcal{J}^\omega(u)$ satisfies the following random equation:

$$\begin{cases} i\partial_t w = \partial_x^4 w + \mathcal{N}_1^\omega(w) + \mathcal{N}_2^\omega(w) \\ w|_{t=0} = u_0^\omega. \end{cases} \tag{1.44}$$

Here, the first nonlinearity $\mathcal{N}_1^\omega(w)$ is defined by

$$\mathcal{N}_1^\omega(w)(x,t) := \sum_{n\in\mathbb{Z}} e^{inx} \sum_{\Gamma(n)} e^{it\Psi^\omega(\bar{n})} \widehat{w}(n_1,t)\overline{\widehat{w}(n_2,t)}\widehat{w}(n_3,t), \tag{1.45}$$

where $\Gamma(n)$ is as in (1.20) and $\Psi^\omega(\bar{n})$ denotes the random phase function:

$$\Psi^\omega(\bar{n}) := \Psi^\omega(n_1,n_2,n_3,n) = |g_{n_1}(\omega)|^2 - |g_{n_2}(\omega)|^2 + |g_{n_3}(\omega)|^2 - |g_n(\omega)|^2. \tag{1.46}$$

The second nonlinearity $\mathcal{N}_2^\omega(w)$ is defined by

$$\mathcal{N}_2^\omega(w)(x,t) := -\sum_{n\in\mathbb{Z}} e^{inx} \big[|\widehat{w}(n,t)|^2 - |g_n(\omega)|^2\big]\widehat{w}(n,t)\,. \tag{1.47}$$

As we can see, (1.45) and (1.47) are random versions of (1.18) and (1.19). The main advantage of working with this gauged version of the renormalized 4NLS (1.6) lies in the weaker resonant nonlinearity $\big[|\widehat{w}(n)|^2 - |g_n(\omega)|^2\big]\widehat{w}(n)$, which would be eliminated if $\widehat{w}(n) = g_n$. This observation turns out to be crucial in our later analysis.

The Duhamel formulation for the gauged solution w is given by

$$w(t) = S(t)u_0^\omega - i\int_0^t S(t-t')\big[\mathcal{N}_1^\omega(w) + \mathcal{N}_2^\omega(w)\big](t')dt'. \tag{1.48}$$

Now by setting $z_1^\omega = S(t)u_0^\omega$, we see that the residual term

$$v = w - z_1^\omega,$$

satisfies the following Duhamel formulation:

$$v(t) = -i\int_0^t S(t-t')\big[\mathcal{N}_1^\omega(v+z_1^\omega) + \mathcal{N}_2^\omega(v+z_1^\omega)\big](t')dt'. \tag{1.49}$$

A naive approach would be to try to solve the fixed point problem (1.49) by a contraction argument (namely, by the Picard iteration scheme) for v in $L^2(\mathbb{T})$, exploiting randomness. It turns out, however, that this naive approach via a contraction argument does *not* work for our problem. In the following, by partially iterating the Duhamel formulation, we prove convergence in $L^2(\mathbb{T})$ of approximating smooth solutions and construct a solution to (1.49) and hence to (1.44). See Section 4 for more details. We establish the crucial nonlinear estimates (Propositions 4.1 and 4.2) by reducing them to boundedness properties of certain random multilinear functionals of the white noise, whose tail estimates are proved in Appendix A.

Remark 1.12. As it will become clear from the analysis below, there is room to extend our analysis to the fractional NLS with dispersion weaker than the fourth order dispersion. However, this would not introduce any new qualitative phenomenon as compared to the case of the fourth order dispersion and hence we only consider the fourth order NLS in this paper. We also point out that the case of the standard NLS (with the second order dispersion) is out of reach at this point. See the introduction in [25] for a discussion on the criticality of this problem (in the context of the stochastic NLS with additive space-time white noise forcing).

Remark 1.13. (i) In the deterministic setting, Takaoka-Tsutsumi [67] implicitly used a gauge transform analogous to (1.43) in the low regularity study of the modified KdV equation to weak the resonant interaction. This led them to work in the modified $X^{s,b}$-spaces. See also [44]. In our case, the gauge transform $\mathcal{J}^\omega$ is random and hence it leads to the random $X^{s,b}$-spaces. See Subsection A.1. We also point out the work [56] on the use of a gauge transform in the probabilistic context.

(ii) In order to construct the dynamics for the $\alpha = 0$ case, we partially iterate the Duhamel formulation (of the gauged equation) and establish convergence property of smooth approximating solutions. See Section 4. This strategy is close in spirit to the work [52, 65]. In the context of stochastic PDEs, such iteration of a Duhamel formulation appears in the dispersive setting [50, 31] and in the parabolic setting [34, 17, 43]. We also mention [8, 11, 12, 13] on the probabilistic construction of solutions by establishing convergence of smooth solutions. In particular, the recent approach by Bourgain-Bulut [11, 12] relying on the invariance of the truncated Gibbs measures even in the construction of local solutions works well for a power-type nonlinearity with positive regularity but is not suitable to our problem at hand. See [4] for a survey on this method.

1.8. **Organization of the paper.** In Section 2, we introduce the basic notations and list some basic deterministic and probabilistic lemmas. In Section 3, we present the proof of Theorem 2 for $\alpha > 0$. The remaining part of the paper is devoted to handle the $\alpha = 0$ case. In Section 4, we prove Theorem 2, by assuming two key nonlinear estimates (Propositions 4.1 and 4.2). In Section 5, we prove Theorem 3 and then Theorem 1. We present the proofs of Propositions 4.1 and 4.2 in Sections 6 and 7. Appendix A contains the proofs of some probabilistic lemmas.

2. Notations and preliminaries

As in the usual low regularity analysis of dispersive PDEs, an important ingredient will be the Fourier restriction norm method introduced in [5]. Given $s, b \in \mathbb{R}$, define $X^{s,b}(\mathbb{T}\times\mathbb{R})$ as a completion of the test functions under the following norm:

$$\|u\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} = \|\langle n\rangle^s \langle \tau + n^4\rangle^b \widehat{u}(n,\tau)\|_{\ell^2_n L^2_\tau}, \tag{2.1}$$

where $\langle\,\cdot\,\rangle = (1+|\cdot|^2)^{\frac12}$. Recall that $X^{s,b}$ embeds into $C(\mathbb{R}; H^s(\mathbb{T}))$ for $b > \frac12$. Given a time interval $I = [a,b]$, we define the local-in-time version $X^{s,b}_I = X^{s,b}([a,b])$ by setting

$$\|u\|_{X^{s,b}_I} = \inf\big\{\|v\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} : v|_I = u\big\}. \tag{2.2}$$

Note that $X^{s,b}_I$ is a Banach space. When $I = [-\delta,\delta]$, we simply set $X^{s,b,\delta} = X^{s,b}_I$. The local-in-time versions of other function spaces are defined analogously.

For simplicity, we often drop 2π in dealing with the Fourier transforms. If a function f is random, we may use the superscript f^ω to show the dependence on $\omega\in\Omega$.

Let $\eta \in C^\infty_c(\mathbb{R})$ be a smooth non-negative cutoff function supported on $[-2,2]$ with $\eta\equiv 1$ on $[-1,1]$ and set

$$\eta_\delta(t) = \eta(\delta^{-1}t) \tag{2.3}$$

for $\delta > 0$. We also denote by $\chi = \chi_{[-1,1]}$ the characteristic function of the interval $[-1,1]$ and let $\chi_\delta(t) = \chi(\delta^{-1}t) = \chi_{[-\delta,\delta]}(t)$.

Let $\mathbb{Z}_{\geq 0} := \mathbb{Z} \cap [0, \infty)$. Given a dyadic number $N \in 2^{\mathbb{Z}_{\geq 0}}$, let $\mathbf{P}_N$ be the (non-homogeneous) Littlewood-Paley projector onto the (spatial) frequencies $\{n \in \mathbb{Z} : |n| \sim N\}$ such that

$$f = \sum_{\substack{N \geq 1 \\ \text{dyadic}}}^{\infty} \mathbf{P}_N f.$$

Given a non-negative integer $N \in \mathbb{Z}_{\geq 0}$, we also define the Dirichlet projector π_N onto the frequencies $\{|n| \leq N\}$ by setting

$$\pi_N f(x) = \sum_{|n| \leq N} \widehat{f}(n) e^{inx}. \tag{2.4}$$

Moreover, we set

$$\pi_N^{\perp} = \operatorname{Id} - \pi_N. \tag{2.5}$$

By convention, we also set $\pi_{-1}^{\perp} = \operatorname{Id}$.

We use c, C to denote various constants, usually depending only on α and s. If a constant depends on other quantities, we will make it explicit. For two quantities A and B, we use $A \lesssim B$ to denote an estimate of the form $A \leq CB$, where C is a universal constant, independent of particular realization of A or B. Similarly, we use $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$. The notation $A \ll B$ means $A \leq cB$ for some sufficiently small constant c. We also use the notation $a+$ (and $a-$) to denote $a + \varepsilon$ (and $a - \varepsilon$, respectively) for arbitrarily small $\varepsilon > 0$ (this notation is often used when there is an implicit constant which diverges in the limit $\varepsilon \to 0$).

2.1. **Deterministic tools.** Define the phase function $\Phi(\bar{n})$ by

$$\Phi(\bar{n}) = \Phi(n_1, n_2, n_3, n) = n_1^4 - n_2^4 + n_3^4 - n^4. \tag{2.6}$$

Then, the phase function $\Phi(\bar{n})$ admits the following factorization. See [57] for the proof.

Lemma 2.1. *Let $n = n_1 - n_2 + n_3$. Then, we have*

$$\Phi(\bar{n}) = (n_1 - n_2)(n_1 - n)\big(n_1^2 + n_2^2 + n_3^2 + n^2 + 2(n_1 + n_3)^2\big).$$

Recall that by restricting the $X^{s,b}$-spaces onto a small time interval $[-\delta, \delta]$, we can gain a small power of δ (at a slight loss in the modulation).

Lemma 2.2. *Let $s \in \mathbb{R}$ and $b < \frac{1}{2}$. Then, there exists $C = C(b) > 0$ such that*

$$\|\eta_\delta(t) \cdot u\|_{X^{s,b}} + \|\chi_\delta(t) \cdot u\|_{X^{s,b}} \leq C \delta^{\frac{1}{2} - b -} \|u\|_{X^{s, \frac{1}{2}-}}.$$

The proof of Lemma 2.2 is based on the following scaling property: $\widehat{\eta}_\delta(\tau) = \delta \widehat{\eta}(\delta\tau)$, yielding

$$\|\widehat{\eta}_\delta\|_{L^q_\tau} \sim \delta^{\frac{q-1}{q}} \|\widehat{\eta}\|_{L^q_\tau} \lesssim \delta^{\frac{q-1}{q}}, \tag{2.7}$$

for $q \geq 1$. See [21] for details.

Next, we collect the basic linear estimates (see [26]).

Lemma 2.3. *Let $s \in \mathbb{R}$.*

(i) *Given $b \in \mathbb{R}$, there exists $C = C(b) > 0$ such that*

$$\|S(t) u_0\|_{X^{s,b,\delta}} \leq C \|u_0\|_{H^s}$$

for any $0 < \delta \leq 1$.

(ii) *Given $b > \frac{1}{2}$, there exists $C = C(b) > 0$ such that*

$$\bigg\| \int_0^t S(t-t')F(x,t')dt' \bigg\|_{X^{s,b,\delta}} \lesssim \|F\|_{X^{s,b-1,\delta}}$$

for any $\delta > 0$.

The following periodic L^4-Strichartz estimate from [57] also plays an important role:

$$\|u\|_{L^4_{x,t}} \lesssim \|u\|_{X^{0,\frac{5}{16}}}. \tag{2.8}$$

Interpolating (2.8) with $\|u\|_{L^2_{x,t}} = \|u\|_{X^{0,0}}$, we have

$$\|u\|_{L^{3+}_{x,t}} \lesssim \|u\|_{X^{0,\frac{5}{24}+}} \qquad \text{and} \qquad \|u\|_{L^{2+}_{x,t}} \lesssim \|u\|_{X^{0,0+}}. \tag{2.9}$$

We also recall the following lemma on convolutions. See [26] for a proof.

Lemma 2.4. *Let $\alpha > \beta \geq 0$ with $\alpha + \beta > 1$. Then, there exists $C > 0$ such that*

$$\int_{\mathbb{R}} \frac{1}{\langle x-y\rangle^\alpha \langle y\rangle^\beta} dy \leq \frac{C}{\langle x\rangle^\gamma}$$

for any $x \in \mathbb{R}$, where γ is given by

$$\gamma = \begin{cases} \alpha+\beta-1, & \text{if } \alpha < 1,\\ \beta - \varepsilon, & \text{if } \alpha = 1,\\ \beta, & \text{if } \alpha > 1 \end{cases}$$

for any small $\varepsilon > 0$.

Lastly, we state two lemmas related to boundedness properties of products in Sobolev spaces.

Lemma 2.5. *Let $\varepsilon > 0$. Then, there exists $C = C(\varepsilon) > 0$ such that*

$$\|fg\|_{H^{\frac{1}{2}-\varepsilon}(\mathbb{R})} \leq C\|f\|_{H^{\frac{1}{2}+\varepsilon}(\mathbb{R})}\|g\|_{H^{\frac{1}{2}-\frac{\varepsilon}{2}}(\mathbb{R})}.$$

Lemma 2.5 easily follows from standard analysis with Littlewood-Paley decompositions and Bernstein's inequality. We omit details.

Lemma 2.6. *Let $0 \leq b < \frac{1}{2}$. Then, we have*

$$\|\mathbf{1}_{[0,T]} \cdot f\|_{H^b(\mathbb{R})} \lesssim \|f\|_{H^b(\mathbb{R})},$$

uniformly in $T \geq 0$.

See [24] for a classical proof via an interpolation argument. By Plancherel's identity, Lemma 2.6 also follows from the boundedness of the Hilbert transform (on the Fourier side) with an A_2-weight $\langle\tau\rangle^{2b}$, $0 \leq b < \frac{1}{2}$. See [27].

2.2. **Probabilistic estimates.** Next, we state several probabilistic lemmas related to Gaussian random variables. See also Appendix A for further lemmas. In the following, $\{g_n\}_{n\in\mathbb{Z}}$ denotes a family of independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$.

We first start by a well known fact (see for example [48, 21]).

Lemma 2.7. *Let $\varepsilon > 0$. Then, there exist $c, C > 0$ such that*

$$P\Big(\sup_{n\in\mathbb{Z}} \langle n\rangle^{-\varepsilon}|g_n(\omega)| > K\Big) < Ce^{-cK^2}$$

for any $K > 0$. In particular, given $\beta > 0$, by choosing $K = \delta^{-\frac{\beta}{2}}$, we have

$$P\Big(\sup_{n\in\mathbb{Z}} \langle n\rangle^{-\varepsilon}|g_n(\omega)| > \delta^{-\frac{\beta}{2}}\Big) < Ce^{-\frac{1}{\delta^c}}$$

for any $\delta > 0$.

Next, we recall the Wiener chaos estimates. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of independent standard Gaussian random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is the σ-algebra generated by this sequence. Given $k \in \mathbb{Z}_{\geq 0}$, we define the homogeneous Wiener chaoses $\mathcal{H}_k$ to be the closure (under $L^2(\Omega)$) of the span of Fourier-Hermite polynomials $\prod_{n=1}^\infty H_{k_n}(g_n)$, where H_j is the Hermite polynomial of degree j and $k = \sum_{n=1}^\infty k_n$.[9] Then, we have the following Ito-Wiener decomposition:

$$L^2(\Omega, \mathcal{F}, P) = \bigoplus_{k=0}^\infty \mathcal{H}_k.$$

See Theorem 1.1.1 in [47]. We also set

$$\mathcal{H}_{\leq k} = \bigoplus_{j=0}^k \mathcal{H}_j \tag{2.10}$$

for $k \in \mathbb{N}$. For example, the random linear solution z_1^ω defined in (1.22) belongs to $\mathcal{H}_1$ (for each fixed $t \in \mathbb{R}$), while z_3^ω in (1.24) belongs to $\mathcal{H}_{\leq 3}$. As pointed out above, the random resonant solution z^ω defined in (1.34) does *not* belong to $\mathcal{H}_{\leq k}$ for any finite $k \in \mathbb{N}$.

In this setting, we have the following Wiener chaos estimate [66, Theorem I.22]. See also [69, Proposition 2.4].

Lemma 2.8. *Let $k \in \mathbb{N}$. Then, we have*

$$\|X\|_{L^p(\Omega)} \leq (p-1)^{\frac{k}{2}}\|X\|_{L^2(\Omega)}$$

for any finite $p \geq 2$ and any $X \in \mathcal{H}_{\leq k}$.

We also recall the following lemma, which is a consequence of Chebyshev's inequality. See, for example, Lemma 4.5 in [71] and the proof of Lemma 3 in [2].[10]

Lemma 2.9. *Let $k \geq 1$. Suppose that there exists $C_0 > 0$ such that a random variable X satisfies $\|X\|_{L^p(\Omega)} \leq C_0 p^{\frac{k}{2}}$ for any finite $p \geq 2$. Then, there exist $c, C > 0$ such that*

$$P\big(|X| > \lambda\big) \leq Ce^{-cC_0^{-\frac{2}{k}}\lambda^{\frac{2}{k}}}$$

for any $\lambda > 0$.

In probabilistic well-posedness theory, a probabilistic improvement of Strichartz estimates for random linear solutions plays an important role. The following lemma states that a similar estimate also holds for the random resonant solution z^ω defined in (1.34).

[9]This implies that $k_n = 0$ except for finitely many n's.

[10]This corresponds to Lemma 2.3 in the arXiv version.

Lemma 2.10. *Given $\alpha \ge 0$, let z^ω be the solution to the resonant 4NLS* (1.31) *given by* (1.34). *Then, given $p \ge 2$ and $\varepsilon > 0$, there exist $c, C > 0$ such that*

$$P\Big(\|\mathbf{P}_N z^\omega\|_{L^p_{x,t}(\mathbb{T}\times[-\delta,\delta])} > N^{\frac{1}{2}-\alpha+\varepsilon}\Big) < Ce^{-\frac{N^{2\varepsilon}}{\delta^c}} \tag{2.11}$$

for any $\delta > 0$ and dyadic $N \ge 1$.

One way to prove Lemma 2.10 would be to directly apply the Wiener chaos estimate (Lemma 2.8) to the $(2k+1)$-fold products of Gaussian random variables in the series expansion (1.35). See Lemma 2.11 for such a direct approach. In the particular case of Lemma 2.10, we can give a shorter proof by exploiting the invariance of a complex-valued mean-zero Gaussian random variable under the transformation: $g \mapsto e^{it|g|^2}g$; see Lemma 4.2 in [57]. This allows us to avoid higher order products of Gaussian random variables.

Proof of Lemma 2.10. Given $n \in \mathbb{Z}$ and $(x,t) \in \mathbb{T}\times\mathbb{R}$, define $h_n(x,t)$ by

$$h_n(x,t) := e^{i(nx-n^4t)}e^{it\frac{|g_n|^2}{\langle n\rangle^{2\alpha}}}\frac{g_n}{\langle n\rangle^\alpha}.$$

Then, it follows from the rotational invariance of complex-valued Gaussian random variables and Lemma 4.2 in [57] that $h_n(x,t) \sim \mathcal{N}_\mathbb{C}(0, \langle n\rangle^{-2\alpha})$ for each *fixed* $(x,t) \in \mathbb{T}\times\mathbb{R}$.

By Minkowski's integral inequality and Lemma 2.8, we have

$$\begin{aligned}\Big(\mathbb{E}\Big[\|\mathbf{P}_N z^\omega\|^r_{L^p_{x,t}(\mathbb{T}\times[-\delta,\delta])}\Big]\Big)^{\frac{1}{r}} &\le \bigg\|\Big\|\sum_{|n|\sim N} h_n(x,t)\Big\|_{L^r(\Omega)}\bigg\|_{L^p_{x,\delta}}\\ &\lesssim \sqrt{r}\,\bigg\|\Big\|\sum_{|n|\sim N} h_n(x,t)\Big\|_{L^2(\Omega)}\bigg\|_{L^p_{x,\delta}}\\ &\lesssim \sqrt{r}\,\delta^{\frac{1}{p}}N^{\frac{1}{2}-\alpha}\end{aligned}$$

for any $r \ge p$. Then, the desired estimate (2.11) follows from Lemma 2.9. □

Finally, we conclude this section by stating a crucial lemma in studying powers of the random resonant solution z^ω in the multilinear $X^{s,b}$-analysis. This lemma also plays an important role in establishing boundedness properties of certain random multilinear functionals of the white noise (see Lemma 6.1 below), which is a key ingredient for the proof of Theorem 2 when $\alpha = 0$. We present the proof of this lemma in Appendix A.

Lemma 2.11. *Fix a non-empty set $\mathcal{A} \subset \{1,2,3\}$ and $k, k_j \in \mathbb{Z}_{\ge 0}$, $j \in \mathcal{A}$, such that*

$$k = \sum_{j\in\mathcal{A}} k_j. \tag{2.12}$$

Given a (deterministic) sequence $\{c^{\bar{k}}_{n_1,n_2,n_3}\}_{n_1,n_2,n_3\in\mathbb{Z}}$ with $\bar{k} = \{k_j\}_{j\in\mathcal{A}}$, define a sequence $\{\Sigma_n\}_{n\in\mathbb{Z}}$ by setting

$$\Sigma_n = \Sigma_n(\bar{k}) = \frac{1}{\prod_{j\in\mathcal{A}} k_j!}\sum_{(n_1,n_2,n_3)\in\Gamma(n)} c^{\bar{k}}_{n_1,n_2,n_3}\prod_{j\in\mathcal{A}}|g_{n_j}|^{2k_j}g^*_{n_j} \tag{2.13}$$

for $n \in \mathbb{Z}$, where $\Gamma(n)$ is as in (1.20) *and $g^*_{n_j}$ is defined by*

$$g^*_{n_j} = \begin{cases} g_{n_j}, & \text{when } j = 1 \text{ or } 3,\\ \overline{g_{n_j}}, & \text{when } j = 2.\end{cases} \tag{2.14}$$

Then, there exists $C > 0$, independent of k and $k_j \in \mathbb{Z}_{\geq 0}$, $j \in \mathcal{A}$, such that

$$\|\Sigma_n\|_{L^p(\Omega)} \leq C^k (p-1)^{k+\frac{|\mathcal{A}|}{2}} \bigg(\sum_{(n_1,n_2,n_3)\in\Gamma(n)} |c^{\bar{k}}_{n_1,n_2,n_3}|^2 \bigg)^{\frac{1}{2}} \tag{2.15}$$

for all $p \geq 2$ and $n \in \mathbb{Z}$.

3. Local theory, Part 1: $0 < \alpha \leq \frac{1}{2}$

In this section, we present the proof of Theorem 2 when $0 < \alpha \leq \frac{1}{2}$. In particular, we show that the Cauchy problem (1.39) for v is almost surely locally well-posed. More precisely, we show that for each small $\delta > 0$, there exists Ω_δ with $P(\Omega_\delta^c) < Ce^{-\frac{1}{\delta^c}}$ such that, for each $\omega \in \Omega_\delta$, the map Γ^ω defined in (1.40) is a contraction on $B(1)$, where $B(1)$ denotes the ball of radius 1 in $X^{0,\frac{1}{2}+,\delta}$ centered at the origin.

Given v on $\mathbb{T} \times [-\delta, \delta]$, let $\widetilde{v}$ be an extension of v onto $\mathbb{T} \times \mathbb{R}$. By the non-homogeneous linear estimate (Lemma 2.3), we have

$$\begin{aligned}\bigg\| \int_0^t S(t-t')\mathfrak{N}^\omega(v)(t')dt' \bigg\|_{X^{0,\frac{1}{2}+,\delta}} &\leq \bigg\| \eta_\delta(t) \int_0^t S(t-t')\mathfrak{N}^\omega(\widetilde{v})(t')dt' \bigg\|_{X^{0,\frac{1}{2}+}} \\ &\lesssim \|\mathfrak{N}^\omega(\widetilde{v})\|_{X^{0,-\frac{1}{2}+}},\end{aligned}$$

where η_δ is a smooth cutoff on $[-2\delta, 2\delta]$ as in (2.3) and

$$\mathfrak{N}^\omega(v) := \chi_\delta \cdot \big(\mathcal{N}(v + \widetilde{z}^\omega) - \mathcal{N}_2(\widetilde{z}^\omega) \big) \tag{3.1}$$

with an extension $\widetilde{z}^\omega$ of the truncated random linear solution $\chi_\delta \cdot z^\omega$ from $[-\delta, \delta]$ to $\mathbb{R}$. Then, our main goal is to prove that there exists $\Omega_\delta \subset \Omega$ and $\theta > 0$ with $P(\Omega_\delta^c) < Ce^{-\frac{1}{\delta^c}}$ such that

$$\|\mathfrak{N}^\omega(\widetilde{v})\|_{X^{0,-\frac{1}{2}+}} \lesssim \delta^\theta \Big(1 + \|\widetilde{v}\|_{X^{0,\frac{1}{2}+}} \Big)^3 \tag{3.2}$$

for all $\omega \in \Omega_\delta$ and for any extension $\widetilde{v}$ of v. By the definition (2.2) of the local-in-time norm, we then conclude from (3.1) and (3.2) that

$$\bigg\| \int_0^t S(t-t')\mathfrak{N}^\omega(v)(t')dt' \bigg\|_{X^{0,\frac{1}{2}+,\delta}} \lesssim \delta^\theta \Big(1 + \|v\|_{X^{0,\frac{1}{2}+,\delta}} \Big)^3.$$

By the trilinear structure of the nonlinearity, a similar estimate holds for the difference $\Gamma^\omega v_1 - \Gamma^\omega v_2$, allowing us to conclude that Γ^ω is a contraction on $B(1) \subset X^{0,\frac{1}{2}+,\delta}$ for $\omega \in \Omega_\delta$. Note that the claim (1.12) follows from the embedding $X^{0,\frac{1}{2}+,\delta} \subset C([-\delta,\delta]; L^2(\mathbb{T}))$ and Lemma 1.9.

In view of (3.1), in order to prove (3.2), we need to carry out case-by-case analysis on

$$\|\chi_\delta \cdot \mathcal{N}_k(u_1, u_2, u_3)\|_{X^{s,-\frac{1}{2}+}}, \qquad k = 1, 2, \tag{3.3}$$

where u_j is taken to be either of type

(I) rough random resonant part:

$$u_j = \widetilde{z}^\omega, \text{ where } \widetilde{z}^\omega \text{ is } \textit{some} \text{ extension of } \chi_\delta \cdot z^\omega,$$

where z^ω denotes the random resonant solution defined in (1.34),

(II) smoother 'deterministic' nonlinear part:

$$u_j = \widetilde{v}_j, \text{ where } \widetilde{v}_j \text{ is } \textit{any} \text{ extension of } v,$$

except for $u_1 = u_2 = u_3 = \widetilde{z}^\omega$ when $k = 2$ (thanks to the subtraction of $\mathcal{N}_2(\widetilde{z}^\omega)$ in (3.1)).

In the following, we take $\widetilde{z}^\omega = \eta_\delta z^\omega$. It follows from (1.34) that

$$\mathcal{F}(\eta_\delta z^\omega)(n,\tau) = \widehat{\eta}_\delta\Big(\tau + n^4 - \tfrac{|g_n|^2}{\langle n\rangle^{2\alpha}}\Big)\cdot \frac{g_n}{\langle n\rangle^\alpha}. \tag{3.4}$$

Thanks to the sharp cutoff function in (3.3), we may take

$$u_j = \chi_\delta \cdot \widetilde{v}_j \tag{3.5}$$

in (3.3) when u_j is of type (II). We use the expressions $u_j(\text{I})$ (and $u_j(\text{II})$, respectively) to mean that u_j is of type (I) (and of type (II), respectively) in the following. We point out that the most intricate case appears when all u_j's are of type (I) in estimating the non-resonant contribution. In this case, a simple application of the Wiener chaos estimate (Lemma 2.8) is no longer applicable and we need to carefully estimate the contribution from the sum of the products of the $(2k_j+1)$-linear term, $k_j \in \mathbb{N}_0$, $j = 1,2,3$, in (1.35), using Lemma 2.11. See Case (D) in Subsection 3.2.

3.1. **Resonant part $\mathcal{N}_2$.** In this subsection, we estimate the resonant part of the nonlinear estimate (3.2). In particular, we prove

$$\|\chi_\delta \cdot \mathcal{N}_2(u_1,u_2,u_3)\|_{X^{0,-\frac12+}} \lesssim \delta^\theta \prod_{j\in\mathcal{I}} \|\widetilde{v}_j\|_{X^{0,\frac12+}} \tag{3.6}$$

for some $\theta > 0$, outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$, where $\mathcal{N}_2$ is the resonant part of the nonlinearity defined in (1.19), u_j is either of type (I) or (II), *except* for the case when all u_j's are of type (I), and the index set $\mathcal{I}$ is defined by

$$\mathcal{I} = \big\{ j \in \{1,2,3\} : u_j \text{ is of type (II)}\big\}. \tag{3.7}$$

We have

$$\text{LHS of (3.6)} = \bigg\| \frac{1}{\langle \tau + n^4\rangle^{\frac12-}} \int_{\tau = \tau_1 - \tau_2 + \tau_3} \widehat{u}_1(n,\tau_1)\overline{\widehat{u}_2(n,\tau_2)}\widehat{u}_3(n,\tau_3)d\tau_1 d\tau_2\bigg\|_{\ell^2_n L^2_\tau}. \tag{3.8}$$

• **Case (a):** u_j of type (II), $j = 1,2,3$.

By Hölder's inequality with p large ($\frac12 = \frac{1}{2+} + \frac1p$), we have

$$(3.8) \lesssim \sup_n \|\langle \tau + n^4\rangle^{-\frac12+}\|_{L^{2+}_\tau} \bigg\| \int_{\tau = \tau_1 - \tau_2 + \tau_3} \widehat{u}_1(n,\tau_1)\overline{\widehat{u}_2(n,\tau_2)}\widehat{u}_3(n,\tau_3)d\tau_1 d\tau_2\bigg\|_{\ell^2_n L^p_\tau}.$$

By Young's and Hölder's inequalities, $\ell^2_n \subset \ell^6_n$, and Lemma 2.2 with (3.5),

$$\begin{aligned} &\lesssim \prod_{j=1}^3 \|\widehat{u}_j(n,\tau)\|_{\ell^6_n L^{\frac32-}_\tau} \lesssim \prod_{j=1}^3 \|\langle \tau + n^4\rangle^{\frac16+}\widehat{u}_j(n,\tau)\|_{\ell^6_n L^2_\tau} \le \prod_{j=1}^3 \|u_j\|_{X^{0,\frac16+}} \\ &\lesssim \delta^{1-} \prod_{j=1}^3 \|\widetilde{v}_j\|_{X^{0,\frac12+}}. \end{aligned}$$

• **Case (b):** Exactly one u_j of type (I). Say $u_1(\mathrm{I})$, $u_2(\mathrm{II})$, and $u_3(\mathrm{II})$.

By Hölder's inequality (with $p \gg 1$ as before), (3.4), and a change of variables, we have

$$\begin{aligned}
(3.8) &\lesssim \sup_n \|\langle \tau + n^4\rangle^{-\frac{1}{2}+}\|_{L^{2+}_\tau} \\
&\quad \times \bigg\| \langle n\rangle^{-\alpha}|g_n| \int_{\tau=\tau_1-\tau_2+\tau_3} \widehat{\eta}_\delta\Big(\tau_1 + n^4 - \tfrac{|g_n|^2}{\langle n\rangle^{2\alpha}}\Big)\overline{\widehat{u}_2(n,\tau_2)}\widehat{u}_3(n,\tau_3)d\tau_1 d\tau_2\bigg\|_{\ell^2_n L^p_\tau} \\
&\lesssim \big(\sup_n \langle n\rangle^{-\alpha}|g_n|\big)\bigg\| \int_{\tau=\zeta_1-\tau_2+\tau_3-C(n,\omega)} \widehat{\eta}_\delta(\zeta_1)\overline{\widehat{u}_2(n,\tau_2)}\widehat{u}_3(n,\tau_3)d\zeta_1 d\tau_2\bigg\|_{\ell^2_n L^p_\tau},
\end{aligned}$$

where $C(n,\omega)$ is defined by

$$C(n,\omega) := n^4 - \tfrac{|g_n|^2}{\langle n\rangle^{2\alpha}}. \tag{3.9}$$

Note that for fixed $n \in \mathbb{Z}$ and $\omega \in \Omega$, $C(n,\omega)$ is a fixed number. Hence, we can apply Young's inequality (in τ, ζ_1, τ_2, and τ_3), Lemma 2.7 with $\beta = 0+$, (2.7), and Lemma 2.2 with (3.5) as above and obtain

$$\begin{aligned}
(3.8) &\lesssim \delta^{\frac{1}{2}-}\big(\sup_n \langle n\rangle^{-\alpha}|g_n|\big)\prod_{j=2}^3 \|\widehat{u}_j(n,\tau)\|_{\ell^4_n L^{\frac{4}{3}}_\tau} \\
&\lesssim \delta^{\frac{1}{2}-}\prod_{j=2}^3 \|\langle \tau + n^4\rangle^{\frac{1}{4}+}\widehat{u}_j(n,\tau)\|_{\ell^4_n L^2_\tau} \le \delta^{\frac{1}{2}-}\prod_{j=2}^3 \|u_j\|_{X^{0,\frac{1}{4}+}} \\
&\lesssim \delta^{1-}\prod_{j=2}^3 \|\widetilde{v}_j\|_{X^{0,\frac{1}{4}+}}
\end{aligned}$$

for any $\alpha > 0$, outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$.

• **Case (c):** Exactly two u_j's of type (I).

First, consider the case $u_1(\mathrm{I})$, $u_2(\mathrm{I})$, and $u_3(\mathrm{II})$. Proceeding as before with $p \gg 1$ and a change of variables, we have

$$\begin{aligned}
(3.8) &\lesssim \bigg\| \langle n\rangle^{-2\alpha}|g_n|^2 \int_{\tau=\tau_1-\tau_2+\tau_3} \widehat{\eta}_\delta\Big(\tau_1 + n^4 - \tfrac{|g_n|^2}{\langle n\rangle^{2\alpha}}\Big) \\
&\qquad\qquad \times \overline{\widehat{\eta}_\delta\Big(\tau_2 + n^4 - \tfrac{|g_n|^2}{\langle n\rangle^{2\alpha}}\Big)}\widehat{u}_3(n,\tau_3)d\tau_1 d\tau_2\bigg\|_{\ell^2_n L^p_\tau} \\
&\le \big(\sup_n \langle n\rangle^{-2\alpha}|g_n|^2\big)\bigg\| \int_{\tau=\zeta_1-\zeta_2+\tau_3} \widehat{\eta}_\delta(\zeta_1)\overline{\widehat{\eta}_\delta(\zeta_2)}\widehat{u}_3(n,\tau_3)d\zeta_1 d\zeta_2\bigg\|_{\ell^2_n L^p_\tau}
\end{aligned}$$

By Lemma 2.7, (2.7), and Lemma 2.2 with (3.5),

$$\begin{aligned}
&\lesssim \delta^{\frac{1}{2}-}\big(\sup_n \langle n\rangle^{-2\alpha}|g_n|^2\big)\|\widehat{u}_3(n,\tau)\|_{\ell^2_n L^2_\tau} \lesssim \delta^{\frac{1}{2}-}\|u_3\|_{X^{0,0}} \\
&\lesssim \delta^{1-}\|\widetilde{v}_3\|_{X^{0,\frac{1}{2}+}}
\end{aligned}$$

for $\alpha > 0$, outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$.

Next, consider the case $u_1(\mathrm{I})$, $u_2(\mathrm{II})$, and $u_3(\mathrm{I})$. Proceeding in a similar manner (with $p \gg 1$ and a change of variables with $C(n,\omega)$ as in (3.9)), we have

$$\begin{aligned}
(3.8) \lesssim \bigg\| \langle n \rangle^{-2\alpha} |g_n|^2 \int_{\tau = \tau_1 - \tau_2 + \tau_3} & \widehat{\eta}_\delta \Big(\tau_1 + n^4 - \tfrac{|g_n|^2}{\langle n \rangle^{2\alpha}} \Big) \\
& \times \overline{\widehat{u}_2(n,\tau_2)} \widehat{\eta}_\delta \Big(\tau_3 + n^4 - \tfrac{|g_n|^2}{\langle n \rangle^{2\alpha}} \Big) d\tau_1 d\tau_2 \bigg\|_{\ell^2_n L^p_\tau} \\
\lesssim \Big(\sup_n \langle n \rangle^{-2\alpha} |g_n|^2 \Big) & \bigg\| \int_{\tau = \zeta_1 - \tau_2 + \zeta_3 - 2C(n,\omega)} \widehat{\eta}_\delta(\zeta_1) \overline{\widehat{u}_2(n,\tau_2)} \widehat{\eta}_\delta(\zeta_3) d\zeta_1 d\zeta_3 \bigg\|_{\ell^2_n L^p_\tau} \\
\lesssim \delta^{\frac{1}{2}-} \Big(\sup_n \langle n \rangle^{-2\alpha} |g_n|^2 \Big) & \| \widehat{u}_2(n,\tau) \|_{\ell^2_n L^2_\tau} \lesssim \delta^{\frac{1}{2}-} \| u_2 \|_{X^{0,0}} \\
\lesssim \delta^{1-} \| \widetilde{v}_2 \|_{X^{0,\frac{1}{2}+}} &
\end{aligned}$$

for $\alpha > 0$, outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$.

3.2. **Non-resonant part $\mathcal{N}_1$.** In this subsection, we evaluate the non-resonant part of the nonlinearity $\mathfrak{N}^\omega(v)$. In particular, we prove

$$\| \chi_\delta \cdot \mathcal{N}_1(u_1,u_2,u_3) \|_{X^{0,-\frac{1}{2}+}} \lesssim \delta^\theta \prod_{j \in \mathcal{I}} \| \widetilde{v}_j \|_{X^{0,\frac{1}{2}+}} \tag{3.10}$$

for some $\theta > 0$, outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$, where $\mathcal{N}_1$ is the non-resonant part of the nonlinearity defined in (1.18), u_j is either of type (I) or (II), and the index set $\mathcal{I}$ is as in (3.7). Set

$$\sigma := \langle \tau + n^4 \rangle \qquad \text{and} \qquad \sigma_j := \langle \tau_j + n_j^4 \rangle, \quad j = 1,2,3,$$

and

$$\sigma_{\max} := \max(\sigma, \sigma_1, \sigma_2, \sigma_3) \qquad \text{and} \qquad n_{\max} := \max\big(|n|, |n_1|, |n_2|, |n_3|\big) + 1. \tag{3.11}$$

Given dyadic numbers $N, N_1, N_2, N_3 \geq 1$, we also set

$$N_{\max} := \max(N, N_1, N_2, N_3).$$

By duality, we can estimate the left-hand side of (3.10) by

$$\sup_{\|w\|_{X^{0,\frac{1}{2}-}} \leq 1} \bigg| \int_{-\delta}^{\delta} \int_{\mathbb{T}} \mathcal{N}_1(u_1,u_2,u_3) \cdot w \, dx dt \bigg|. \tag{3.12}$$

Without loss of generality, we may assume that $w = \chi_\delta \cdot w$.

- **Case (A):** u_j of type (II), $j = 1,2,3$.

  By Hölder's inequality, (2.8), and Lemma 2.2 with (3.5), we have

$$(3.12) \lesssim \prod_{j=1}^3 \| u_j \|_{L^4_{x,t}} \| w \|_{L^4_{x,t}} \lesssim \delta^{\frac{3}{4}-} \prod_{j=1}^3 \| \widetilde{v}_j \|_{X^{0,\frac{1}{2}+}} \| w \|_{X^{0,\frac{1}{2}-}}.$$

- **Case (B):** Exactly one u_j of type (I). Say $u_1(\mathrm{I})$, $u_2(\mathrm{II})$, and $u_3(\mathrm{II})$.

  First suppose that $\max(\sigma_2, \sigma_3, \sigma) \sim \sigma_{\max}$. Then, it follows from Lemma 2.1 that

$$\max(\sigma_2, \sigma_3, \sigma)^{\frac{7}{24}-} \sim \sigma_{\max}^{\frac{7}{24}-} \gtrsim N_{\max}^{\frac{7}{12}-}. \tag{3.13}$$

By $L^p_{x,t}L^{3+}_{x,t}L^{3+}_{x,t}L^{3+}_{x,t}$-Hölder's inequality with p large, (2.9), Lemma 2.10, Lemma 2.2, and (3.13), we have

$$\begin{aligned}
(3.12) &\lesssim \sum_{\substack{N,N_1,N_2,N_3\\ \text{dyadic}}} \|\mathbf{P}_{N_1}u_1\|_{L^p_{x,t}} \|\mathbf{P}_{N_2}u_2\|_{X^{0,\frac{5}{24}+}} \|\mathbf{P}_{N_3}u_3\|_{X^{0,\frac{5}{24}+}} \|\mathbf{P}_N w\|_{X^{0,\frac{5}{24}+}}\\
&\lesssim \sum_{\substack{N,N_1,N_2,N_3\\ \text{dyadic}}} N_1^{\frac{1}{2}-\alpha+} \|\mathbf{P}_{N_2}u_2\|_{X^{0,\frac{5}{24}+}} \|\mathbf{P}_{N_3}u_3\|_{X^{0,\frac{5}{24}+}} \|\mathbf{P}_N w\|_{X^{0,\frac{5}{24}+}}\\
&\lesssim \delta^{\frac{7}{12}-} \sum_{\substack{N,N_1,N_2,N_3\\ \text{dyadic}}} N_{\max}^{-\frac{1}{12}+} \|\mathbf{P}_{N_2}\widetilde{v}_2\|_{X^{0,\frac{1}{2}+}} \|\mathbf{P}_{N_3}\widetilde{v}_3\|_{X^{0,\frac{1}{2}+}} \|\mathbf{P}_N w\|_{X^{0,\frac{1}{2}-}}\\
&\lesssim \delta^{\frac{7}{12}-} \prod_{j=2}^{3} \|\widetilde{v}_j\|_{X^{0,\frac{1}{2}+}} \qquad (3.14)
\end{aligned}$$

for $\alpha \geq 0$, outside an exceptional set of probability

$$< \sum_{\substack{N_1\geq 1\\ \text{dyadic}}} Ce^{-\frac{N_1^{\varepsilon}}{\delta^c}} \lesssim e^{-\frac{1}{\delta^c}}.$$

Next, suppose that $\max(\sigma_2,\sigma_3,\sigma) \ll \sigma_{\max}$, namely $\sigma_1 \sim \sigma_{\max}$. We first consider the case $\delta^\beta \gg N_{\max}^{-2+2\varepsilon}$ for some small $\beta,\varepsilon > 0$. It follows from Lemmas 2.1 and 2.7 that there exists a set $\Omega_{\beta,\varepsilon} \subset \Omega$ with $P(\Omega^c_{\beta,\varepsilon}) < Ce^{-\frac{1}{\delta^c}}$ such that

$$\frac{|g_{n_1}|^2}{\langle n_1\rangle^{2\alpha}} \lesssim \delta^{-\beta}\langle n_1\rangle^{\varepsilon} \ll N_{\max}^{2-\varepsilon} \ll \sigma_{\max},$$

on $\Omega_{\beta,\varepsilon}$, uniformly in $n_1 \in \mathbb{Z}$, as long as $\alpha \geq 0$. Hence, we have

$$\Big|\widehat{\eta}_\delta\Big(\tau_1 + n_1^4 - \tfrac{|g_{n_1}|^2}{\langle n_1\rangle^{2\alpha}}\Big)\Big| \lesssim \frac{1}{\sigma_1} \lesssim \frac{1}{N_{\max}^2|(n-n_1)(n-n_3)|} \qquad (3.15)$$

on $\Omega_{\beta,\varepsilon}$. Then, by Hölder's inequality (with $p \gg 1$ as in Case (a)), (3.4), (3.15), Young's inequality, and Lemma 2.7 (with $\beta \ll 1$), the contribution to (3.12) in this case is bounded by

$$\begin{aligned}
&\lesssim \sum_{\substack{N,N_1,N_2,N_3,\text{dyadic}\\ \delta^\beta \gg N_{\max}^{-2+2\varepsilon}}} \Bigg\| \sum_{\substack{(n_1,n_2,n_3)\in\Gamma(n)\\ |n|\sim N, |n_j|\sim N_j}} \frac{|g_{n_1}|}{\langle n_1\rangle^{\alpha}} \frac{1}{\big\{N_{\max}^2|(n-n_1)(n-n_3)|\big\}^{\frac{1}{2}+\varepsilon}}\\
&\quad \times \int_{\tau=\tau_1-\tau_2+\tau_3} \Big|\widehat{\eta}_\delta\Big(\tau_1+n_1^4-\tfrac{|g_{n_1}|^2}{\langle n_1\rangle^{2\alpha}}\Big)\Big|^{\frac{1}{2}-\varepsilon} |\widehat{\mathbf{P}_{N_2}u}_2(n_2,\tau_2)||\widehat{\mathbf{P}_{N_3}u}_3(n_3,\tau_3)|d\tau_1 d\tau_2 \Bigg\|_{\ell^2_n L^p_\tau}\\
&\lesssim \big(\sup_{n_1}\langle n_1\rangle^{-\alpha-\varepsilon}|g_{n_1}|\big) \sum_{\substack{N,N_1,N_2,N_3,\text{dyadic}\\ \delta^\beta \gg N_{\max}^{-2+2\varepsilon}}} \Bigg\| \sum_{\substack{(n_1,n_2,n_3)\in\Gamma(n)\\ |n|\sim N, |n_j|\sim N_j}} \frac{1}{\big\{N_{\max}^2|(n-n_1)(n-n_3)|\big\}^{\frac{1}{2}+\frac{\varepsilon}{2}}}\\
&\quad \times \big\||\widehat{\eta}_\delta|^{\frac{1}{2}-\varepsilon}\big\|_{L_\tau^{\frac{p}{3}}} \prod_{j=2}^{3} \|\widehat{\mathbf{P}_{N_j}u}_j(n_j,\tau_j)\|_{L_{\tau_j}^{\frac{p}{p-1}}} \Bigg\|_{\ell^2_n}\\
&\lesssim \delta^{\frac{1}{2}-\varepsilon-\frac{3}{p}-\frac{\beta}{2}} \sum_{\substack{N,N_1,N_2,N_3,\text{dyadic}\\ \delta^\beta \gg N_{\max}^{-2+2\varepsilon}}} N_{\max}^{0-} \prod_{j=2}^{3} \|\widehat{\mathbf{P}_{N_j}u}_j(n_j,\tau_j)\|_{\ell^2_{n_j} L_{\tau_j}^{\frac{p}{p-1}}}
\end{aligned}$$

$$\lesssim \delta^{\frac{1}{2}-\varepsilon-\frac{3}{p}-\frac{\beta}{2}} \prod_{j=2}^{3} \|\widetilde{v}_j\|_{X^{0,\frac{1}{2}+}}$$

for $\alpha \geq 0$, outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$.

Lastly, we consider the case $\delta^\beta \lesssim N_{\max}^{-2+2\varepsilon}$. Proceeding as in (3.14), we bound the contribution of this case to (3.12) by

$$\begin{aligned}
&\lesssim \delta^{\frac{21}{24}-} \sum_{\substack{N,N_1,N_2,N_3\\ \text{dyadic}}} N_1^{\frac{1}{2}-\alpha+} \|\mathbf{P}_{N_2}\widetilde{v}_2\|_{X^{0,\frac{1}{2}+}} \|\mathbf{P}_{N_3}\widetilde{v}_3\|_{X^{0,\frac{1}{2}+}} \|\mathbf{P}_N w\|_{X^{0,\frac{1}{2}-}}\\
&\lesssim \delta^{\frac{21}{24}-\beta-} \sum_{\substack{N,N_1,N_2,N_3\\ \text{dyadic}}} N_{\max}^{-\frac{3}{2}+} \|\mathbf{P}_{N_2}\widetilde{v}_2\|_{X^{0,\frac{1}{2}+}} \|\mathbf{P}_{N_3}\widetilde{v}_3\|_{X^{0,\frac{1}{2}+}}\\
&\lesssim \delta^{\frac{21}{24}-\beta-} \prod_{j=2}^{3} \|\widetilde{v}_j\|_{X^{0,\frac{1}{2}+}}
\end{aligned}$$

for $\alpha \geq 0$, outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$.

- **Case (C):** Exactly two u_j's of type (I). Say $u_1(\mathrm{I})$, $u_2(\mathrm{I})$, and $u_3(\mathrm{II})$.

First, suppose that $\max(\sigma_3, \sigma) \sim \sigma_{\max}$. Then, it follows from Lemma 2.1 that

$$\max(\sigma_3, \sigma)^{\frac{1}{2}-} \sim \sigma_{\max}^{\frac{1}{2}-} \gtrsim N_{\max}^{1-}. \tag{3.16}$$

Suppose that $\sigma \sim \sigma_{\max}$. Then, by $L^p_{x,t}L^p_{x,t}L^{2+}_{x,t}L^2_{x,t}$-Hölder's inequality with p large, (2.9), Lemma 2.10, Lemma 2.2, and (3.16), we have

$$\begin{aligned}
(3.12) &\lesssim \sum_{\substack{N,N_1,N_2,N_3\\ \text{dyadic}}} \|\mathbf{P}_{N_1}u_1\|_{L^p_{x,t}} \|\mathbf{P}_{N_2}u_2\|_{L^p_{x,t}} \|\mathbf{P}_{N_3}u_3\|_{X^{0,0+}} \|\mathbf{P}_N w\|_{X^{0,0}}\\
&\lesssim \sum_{\substack{N,N_1,N_2,N_3\\ \text{dyadic}}} N_1^{\frac{1}{2}-\alpha+} N_2^{\frac{1}{2}-\alpha+} \|\mathbf{P}_{N_3}u_3\|_{X^{0,0+}} \|\mathbf{P}_N w\|_{X^{0,0}}\\
&\lesssim \delta^{\frac{1}{2}-} \sum_{\substack{N,N_1,N_2,N_3\\ \text{dyadic}}} N_{\max}^{-2\alpha+} \|\mathbf{P}_{N_3}\widetilde{v}_3\|_{X^{0,\frac{1}{2}+}} \|\mathbf{P}_N w\|_{X^{0,\frac{1}{2}-}}\\
&\lesssim \delta^{\frac{1}{2}-} \|\widetilde{v}_3\|_{X^{0,\frac{1}{2}+}}
\end{aligned}$$

for $\alpha > 0$, outside an exceptional set of probability

$$< \sum_{\substack{N_1\geq 1\\ \text{dyadic}}} Ce^{-\frac{N_1^\varepsilon}{\delta^c}} + \sum_{\substack{N_2\geq 1\\ \text{dyadic}}} Ce^{-\frac{N_2^\varepsilon}{\delta^c}} \lesssim e^{-\frac{1}{\delta^c}}.$$

A similar argument holds when $\sigma_3 \sim \sigma_{\max}$.

Next, suppose that $\max(\sigma_3, \sigma) \ll \sigma_{\max}$, namely $\max(\sigma_1, \sigma_2) \sim \sigma_{\max}$. Without loss of generality, suppose that $\sigma_1 \sim \sigma_{\max}$. We first consider the case $\delta^\beta \gg N_{\max}^{-2+2\varepsilon}$ for some small $\beta, \varepsilon > 0$.

Proceeding as in Case (B) above, the contribution to (3.12) is bounded by

$$
\begin{aligned}
&\lesssim \sum_{\substack{N,N_1,N_2,N_3,\text{dyadic}\\ \delta^\beta \gg N_{\max}^{-2+2\varepsilon}}} \bigg\| \sum_{\substack{(n_1,n_2,n_3)\in\Gamma(n)\\ |n|\sim N, |n_1|\sim N_1}} \bigg(\prod_{j=1}^{2} \frac{|g_{n_j}|}{\langle n_j\rangle^\alpha}\bigg) \frac{1}{\big\{N_{\max}^2 |(n-n_1)(n-n_3)|\big\}^{\frac12+\varepsilon}} \\
&\quad\times \int_{\tau=\tau_1-\tau_2+\tau_3} \Big|\widehat{\eta}_\delta\Big(\tau_1+n_1^4-\tfrac{|g_{n_1}|^2}{\langle n_1\rangle^{2\alpha}}\Big)\Big|^{\frac12-\varepsilon} \Big|\widehat{\eta}_\delta\Big(\tau_2+n_2^4-\tfrac{|g_{n_2}|^2}{\langle n_2\rangle^{2\alpha}}\Big)\Big| |\widehat{\mathbf{P}_{N_3}u_3}(n_3,\tau_3)| d\tau_1 d\tau_2 \bigg\|_{\ell^2_n L^p_\tau} \\
&\lesssim \bigg(\prod_{j=1}^{2}\sup_{n_j}\langle n_j\rangle^{-\alpha-\frac{\varepsilon}{2}}|g_{n_j}|\bigg) \sum_{\substack{N,N_1,N_2,N_3,\text{dyadic}\\ \delta^\beta \gg N_{\max}^{-2+2\varepsilon}}} \bigg\| \sum_{\substack{(n_1,n_2,n_3)\in\Gamma(n)\\ |n|\sim N, |n_1|\sim N_1}} \frac{1}{\big\{N_{\max}^2 |(n-n_1)(n-n_3)|\big\}^{\frac12+\frac{\varepsilon}{2}}} \\
&\quad\times \Big\| |\widehat{\eta}_\delta|^{\frac12-\varepsilon}\Big\|_{L_\tau^{\frac{p}{2}}} \|\widehat{\eta}_\delta\|_{L^1_\tau} \|\widehat{\mathbf{P}_{N_3}u_3}(n_3,\tau_3)\|_{L_{\tau_3}^{\frac{p}{p-1}}} \bigg\|_{\ell^2_n} \\
&\lesssim \delta^{\frac12-\varepsilon-\frac{2}{p}-\beta} \sum_{\substack{N,N_1,N_2,N_3,\text{dyadic}\\ \delta^\beta \gg N_{\max}^{-2+2\varepsilon}}} N_{\max}^{0-} \|\widehat{\mathbf{P}_{N_3}u_3}(n_3,\tau_3)\|_{\ell^2_n L_{\tau_3}^{\frac{p}{p-1}}} \\
&\lesssim \delta^{\frac12-\varepsilon-\frac{2}{p}-\beta} \|\widetilde{v}_3\|_{X^{0,\frac12}}
\end{aligned}
$$

for $\alpha \geq 0$, outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$.

Lastly, we consider the case $\delta^\beta \lesssim N_{\max}^{-2+2\varepsilon}$. Proceeding as in (3.14) but with $L^p_{x,t}L^p_{x,t}L^{2+}_{x,t}L^2_{x,t}$-Hölder's inequality, the contribution of this case to (3.12)

$$
\begin{aligned}
&\lesssim \sum_{\substack{N,N_1,N_2,N_3\\ \text{dyadic}}} N_{\max}^{1-2\alpha+} \|\mathbf{P}_{N_3}u_3\|_{X^{0,0+}} \|\mathbf{P}_N w\|_{X^{0,0}} \\
&\lesssim \delta^{1-\beta-} \sum_{\substack{N,N_1,N_2,N_3\\ \text{dyadic}}} N_{\max}^{-1-2\alpha+} \|\mathbf{P}_{N_3}\widetilde{v}_3\|_{X^{0,\frac12+}} \|\mathbf{P}_N w\|_{X^{0,\frac12-}} \\
&\lesssim \delta^{1-\beta-} \|\widetilde{v}_3\|_{X^{0,\frac12+}}
\end{aligned}
$$

for $\alpha \geq 0$, outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$.

- **Case (D):** u_j of type (I), $j = 1, 2, 3$.

Fix small $\delta > 0$ (to be chosen later). From (1.35) and (2.3), we have

$$
\mathcal{F}(\eta_\delta z^\omega)(n,\tau) = \delta \sum_{k=0}^{\infty} \frac{(-\delta)^k}{k!} (\partial^k \widehat{\eta})(\delta(\tau+n^4)) \frac{|g_n|^{2k} g_n}{\langle n\rangle^{(2k+1)\alpha}}\,.
$$

Then, we have

$$
\begin{aligned}
\|\mathcal{N}_1(\eta_\delta z^\omega)\|_{X^{0,-\frac12+}} = \bigg\| \frac{1}{\langle \tau+n^4\rangle^{\frac12-}} \sum_{k=0}^{\infty} \sum_{\substack{k_1,k_2,k_3=0\\ k=k_1+k_2+k_3}}^{\infty} \frac{(-\delta)^k}{k_1!k_2!k_3!} \\
\times \sum_{(n_1,n_2,n_3)\in\Gamma(n)} c^{k_1,k_2,k_3}_{n_1,n_2,n_3}(\tau,\delta) \prod_{j=1}^{3} |g_{n_j}|^{2k_j} g^*_{n_j} \bigg\|_{\ell^2_n L^2_\tau}, \qquad (3.17)
\end{aligned}
$$

where $g^*_{n_j}$ is as in (2.14) and $c^{k_1,k_2,k_3}_{n_1,n_2,n_3}(\tau,\delta)$ is defined by

$$c^{k_1,k_2,k_3}_{n_1,n_2,n_3}(\tau,\delta) = \delta^3 \int_{\tau=\tau_1-\tau_2+\tau_3} \prod_{j=1}^{3} \frac{(\partial^{k_j}\widehat{\eta}_j)(\delta(\tau_j+n_j^4))}{\langle n_j\rangle^{(2k_j+1)\alpha}} d\tau_1 d\tau_2$$

with the convention that $\widehat{\eta}_j = \widehat{\eta}$ when $j=1$ or 3 and $\widehat{\eta}_j = \overline{\widehat{\eta}}$ when $j=2$. Then, by Minkowski's integral inequality and Lemma 2.11, there exists $C>0$ such that

$$\begin{aligned} \big\| \|\mathcal{N}_1(\eta_\delta z^\omega)\|_{X^{0,-\frac{1}{2}+}}\big\|_{L^p(\Omega)} & \\ \le p^{\frac{3}{2}} \sum_{k=0}^{\infty} \sum_{\substack{k_1,k_2,k_3=0\\ k=k_1+k_2+k_3}}^{\infty} & (Cp\delta)^k \\ & \times \bigg(\int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \sum_{(n_1,n_2,n_3)\in\Gamma(n)} \frac{1}{\langle \tau+n^4\rangle^{1-}} |c^{k_1,k_2,k_3}_{n_1,n_2,n_3}(\tau,\delta)|^2 d\tau\bigg)^{\frac{1}{2}} \end{aligned} \tag{3.18}$$

for any $p\ge 2$. In the following, we estimate (3.18) with

$$p = \delta^{-\theta} \gg 1 \tag{3.19}$$

for some sufficiently small $\theta>0$. Note that, from Lemma 2.1 and $n\ne n_1,n_3$, we have

$$\sigma_{\max} \gtrsim n_{\max}^2 |(n-n_1)(n-n_3)| \ge n^2_{\max}. \tag{3.20}$$

$\circ$ Subcase (D.1): $\sigma\sim\sigma_{\max}$. First, note that, in view of $\operatorname{supp}\eta\subset[-2,2]$, we have

$$|\mathcal{F}^{-1}(\partial^{k_j}\widehat{\eta})(t)| = |(-it)^{k_j}\eta(t)| \le C^{k_j}\eta(t)\,. \tag{3.21}$$

Then, by a change of variables: $\zeta = \delta\tau + n_1^4 - n_2^4 + n_3^4$ and $\zeta_j = \delta(\tau_j + n_j^4)$, $j=1,2,3$, Plancherel's identity, Hölder's inequality (in t) with (3.21), and $k=k_1+k_2+k_3$, we have

$$\begin{aligned} \|c^{k_1,k_2,k_3}_{n_1,n_2,n_3}(\tau,\delta)\|_{L^2_\tau} &= \delta^{\frac{1}{2}} \bigg\| \int_{\zeta=\zeta_1-\zeta_2+\zeta_3} \prod_{j=1}^{3} \frac{\partial^{k_j}\widehat{\eta}_j(\zeta_j)}{\langle n_j\rangle^{(2k_j+1)\alpha}} d\zeta_1 d\zeta_2 \bigg\|_{L^2_\zeta} \\ &= \delta^{\frac{1}{2}} \bigg\| \prod_{j=1}^{3} \frac{\mathcal{F}^{-1}(\partial^{k_j}\widehat{\eta}_j)}{\langle n_j\rangle^{(2k_j+1)\alpha}} \bigg\|_{L^2_t} \\ &\le C^k \delta^{\frac{1}{2}} \prod_{j=1}^{3} \frac{1}{\langle n_j\rangle^{(2k_j+1)\alpha}}. \end{aligned} \tag{3.22}$$

From (3.18), (3.20) and (3.22), we bound the contribution to

$$\big\| \|\mathcal{N}_1(\eta_\delta z^\omega)\|_{X^{0,-\frac{1}{2}+}}\big\|_{L^p(\Omega)}$$

in this case by

$$\begin{aligned} p^{\frac{3}{2}}\delta^{\frac{1}{2}} \sum_{k_1,k_2,k_3=0}^{\infty} & (Cp\delta)^{k_1}(Cp\delta)^{k_2}(Cp\delta)^{k_3} \\ & \times \bigg(\sum_{n\in\mathbb{Z}} \sum_{(n_1,n_2,n_3)\in\Gamma(n)} \frac{1}{\{n^2_{\max}(n-n_1)(n-n_3)\}^{1-}} \prod_{j=1}^{3} \frac{1}{\langle n_j\rangle^{(2k_j+1)\alpha}}\bigg)^{\frac{1}{2}} \end{aligned}$$

By choosing small $\delta = \delta(C) > 0$ such that $Cp\delta = C\delta^{1-\theta} < 1$,

$$\lesssim p^{\frac{3}{2}} \delta^{\frac{1}{2}} \tag{3.23}$$

for $\alpha \geq 0$.

$\circ$ Subcase (D.2): $\sigma \ll \sigma_{\max}$. Assume that $\sigma_1 \sim \sigma_{\max}$. A similar argument holds when $\sigma_2 \sim \sigma_{\max}$ or $\sigma_3 \sim \sigma_{\max}$.

From (3.17), Hölder's inequality with q large $\big(\frac{1}{2} = \frac{1}{2+} + \frac{1}{q}\big)$, Minkowski's integral inequality, and Lemma 2.11, we have

$$\begin{aligned} &\Big\| \|\mathcal{N}_1(\eta_\delta z^\omega)\|_{X^{0,-\frac{1}{2}+}} \Big\|_{L^p(\Omega)} \\ &\qquad \lesssim p^{\frac{3}{2}} \sum_{k=0}^{\infty} \sum_{\substack{k_1,k_2,k_3=0 \\ k=k_1+k_2+k_3}}^{\infty} (Cp\delta)^k \Bigg(\sum_{n\in\mathbb{Z}} \sum_{\substack{n=n_1-n_2+n_3 \\ n_2 \neq n_1,n_3}} \|c^{k_1,k_2,k_3}_{n_1,n_2,n_3}(\tau,\delta)\|^2_{L^q_\tau} \Bigg)^{\frac{1}{2}} \end{aligned} \tag{3.24}$$

for any $p \geq q$. By integration by parts, we have

$$|\partial^{k_1} \widehat{\eta}(\tau)| = \bigg| \frac{1}{|\tau|^\beta} \int \frac{d^\beta}{dt^\beta} \big(t^{k_1}\eta(t)\big) e^{it\tau} dt \bigg|$$

for $\tau \neq 0$. In particular, with $\beta = 1$, we have

$$\|\partial^{k_1} \widehat{\eta}_1(\tau)\|_{L_\tau^{\frac{2q}{q+2}}(|\tau|\gtrsim K)} \lesssim \frac{C^{k_1}}{K^{1-\frac{q+2}{2q}}}. \tag{3.25}$$

By a change of variables (as in (3.22)) and Young's inequality, (3.25) with $K \sim \delta\sigma_1$, (3.20), and (3.21), we can bound the contribution to $\|c^{k_1,k_2,k_3}_{n_1,n_2,n_3}(\tau,\delta)\|_{L^q_\tau}$ in this case by

$$\begin{aligned} &\delta^{1-\frac{1}{q}} \bigg(\prod_{j=1}^{3} \frac{1}{\langle n_j \rangle^{(2k_j+1)\alpha}} \bigg) \|\partial^{k_1}\widehat{\eta}_1(\tau)\|_{L_\tau^{\frac{2q}{q+2}}(|\tau|\gtrsim K)} \big\| \mathcal{F}^{-1}(\partial^{k_2}\widehat{\eta}_2) \mathcal{F}^{-1}(\partial^{k_3}\widehat{\eta}_3) \big\|_{L^2_t} \\ &\leq C^k \delta^{\frac{1}{2}} \prod_{j=1}^{3} \frac{1}{\langle n_j \rangle^{(2k_j+1)\alpha}} \frac{1}{\big\{ n^2_{\max}(n-n_1)(n-n_3) \big\}^{1-\frac{q+2}{2q}}}. \end{aligned} \tag{3.26}$$

Hence, by choosing $q \gg 1$ and proceeding as in (3.23), we conclude from (3.24) and (3.26) that the contribution to $\Big\| \|\mathcal{N}_1(\eta_\delta z^\omega)\|_{X^{0,-\frac{1}{2}+}} \Big\|_{L^p(\Omega)}$ in this case is also bounded by

$$\lesssim p^{\frac{3}{2}} \delta^{\frac{1}{2}}. \tag{3.27}$$

Finally, by Chebyshev's inequality with (3.23) and (3.27), we have

$$P\Big(\|\mathcal{N}_1(\eta_\delta z^\omega)\|_{X^{0,-\frac{1}{2}+}} > \lambda \Big) \leq C^p \lambda^{-p} p^{\frac{3}{2}p} \delta^{\frac{p}{2}}$$

for any $\lambda > 0$. Letting $\lambda = Cp^2\delta^{\frac{1}{2}}$ and $p = \delta^{-\theta}$ as in (3.19), we have

$$P\Big(\|\mathcal{N}_1(\eta_\delta z^\omega)\|_{X^{0,-\frac{1}{2}+}} > C\delta^{\frac{1}{2}-2\theta} \Big) \leq e^{-p\ln\sqrt{p}} \leq e^{-\frac{1}{\delta^c}}$$

for all $\alpha \geq 0$. In other words, we have

$$\|\mathcal{N}_1(\eta_\delta z^\omega)\|_{X^{0,-\frac{1}{2}+}} \leq C\delta^{\frac{1}{2}-}$$

for $\alpha \geq 0$, outside an exceptional set of probability $\lesssim e^{-\frac{1}{\delta^c}}$.

This completes the proof of the nonlinear estimate (3.2) and hence the proof of Theorem 2 for $0 < \alpha \leq \frac{1}{2}$.

4. Local theory, Part 2: $\alpha = 0$

The remaining part of this paper is devoted to the $\alpha = 0$ case. Namely, we consider the white noise initial data. In this section, we present the proof of almost sure local well-posedness (Theorem 2) by establishing convergence of smooth approximating solutions. The key ingredients are Propositions 4.1 and 4.2, whose proofs will be presented in Sections 6 and 7, respectively.

4.1. **Partially iterated Duhamel formulation.** In Section 1, we introduced the random gauge transform $\mathcal{J}^\omega$ in (1.42) and converted the renormalized 4NLS (1.6) into the random equation (1.44) for $w = \mathcal{J}^\omega(u)$. In the following, we study the Duhamel formulation (1.48) for this random equation. Define

$$\begin{aligned}\mathcal{I}_1(w_1, w_2, w_3)(t) &:= -i\int_0^t S(t-t')\mathcal{N}_1^\omega(w_1, w_2, w_3)(t')dt',\\ \mathcal{I}_2(w)(t) &:= -i\int_0^t S(t-t')\mathcal{N}_2^\omega(w)(t')dt',\end{aligned} \tag{4.1}$$

where $\mathcal{N}_1^\omega(w_1, w_2, w_3)$ is defined by

$$\mathcal{N}_1^\omega(w_1, w_2, w_3)(x,t) := \sum_{n\in\mathbb{Z}} e^{inx}\sum_{\Gamma(n)} e^{it\Psi^\omega(\bar{n})}\widehat{w}_1(n_1,t)\overline{\widehat{w}_2(n_2,t)}\widehat{w}_3(n_3,t)$$

with the random phase function Ψ^ω defined in (1.46) and $\mathcal{N}_2^\omega(w)$ is as in (1.47). By setting $\mathcal{I}_1(w) := \mathcal{I}_1(w,w,w)$, we define $\mathcal{I}(w) := \mathcal{I}_1(w) + \mathcal{I}_2(w)$. Then, we can write the Duhamel formulation (1.48) for $w = \mathcal{J}^\omega(u)$ as

$$w = S(t)u_0^\omega + \mathcal{I}(w), \tag{4.2}$$

If we were to apply the strategy for the $\alpha > 0$ case discussed in Section 3, then by noting that $\mathcal{J}^\omega(z^\omega) = S(t)u_0^\omega$, we would write $v = w - S(t)u_0^\omega$ and try to solve the fixed point problem for v:

$$v = \mathcal{I}_1(v + S(t)u_0^\omega) + \mathcal{I}_2(v + S(t)u_0^\omega) \tag{4.3}$$

by a contraction argument. As mentioned in Section 1, however, we are not able to solve the fixed point problem (4.3) by a contraction argument. In the following, we reformulate the equation by *assuming* that w is a solution to (4.2) and study the reformulated problem. Recalling that $\widehat{w}(n,0) = g_n$ and that w satisfies the equation (1.44), we formally have

$$\begin{aligned}|\widehat{w}(n,t)|^2 - |g_n|^2 &= \int_0^t \frac{d}{dt}|w(n,t')|^2dt'\\ &= -2\,\mathrm{Re}\, i\int_0^t \sum_{\Gamma(n)} e^{it\Psi^\omega(\bar{n})}\widehat{w}(n_1,t')\overline{\widehat{w}(n_2,t')}\widehat{w}(n_3,t')\overline{\widehat{w}(n,t')}dt'\\ &=: \mathcal{E}_n(w,w,w,w)(t).\end{aligned} \tag{4.4}$$

In view of (1.47), (4.1), and (4.4), we then have

$$\mathcal{I}_2(w) = i\int_0^t S(t-t')\sum_{n\in\mathbb{Z}} e^{inx}\mathcal{E}_n(w,w,w,w)(t')\widehat{w}(n,t')dt'$$

for a solution w to (1.44). We denote by $\widetilde{\mathcal{I}}_2(w)$ the quintilinear operator $\widetilde{\mathcal{I}}_2^\omega(w,w,w,w,w)$ given by

$$\widetilde{\mathcal{I}}_2^\omega(w_1,w_2,w_3,w_4,w_5)(x,t) := \int_0^t S(t-t')\sum_{n\in\mathbb{Z}} e^{inx}\mathcal{E}_n(w_1,w_2,w_3,w_4)(t')\widehat{w}_5(n,t')dt'.$$

Then, for a solution w to (1.44), the equality

$$\mathcal{I}_2(w) = \widetilde{\mathcal{I}}_2(w) \tag{4.5}$$

formally holds. As a result, we can rewrite (4.2) as the following partially iterated Duhamel formulation with cubic and quintic nonlinearities:

$$w = S(t)u_0^\omega + \mathcal{I}_1(w) + \widetilde{\mathcal{I}}_2(w). \tag{4.6}$$

We then obtain the following fixed point problem for $v = w - S(t)u_0^\omega$:

$$v = \mathcal{I}_1(v + S(t)u_0^\omega) + \widetilde{\mathcal{I}}_2(v + S(t)u_0^\omega). \tag{4.7}$$

It turns out that the quintic term $\widetilde{\mathcal{I}}_2(v + S(t)u_0^\omega)$ has a better regularity property than the original cubic resonant nonlinearity $\mathcal{I}_2(v + S(t)u_0^\omega)$, which enables us to solve the fixed point problem (4.7) for v by a contraction argument. See Remark 4.4 below. Note, however, that in deriving the equation (4.7), we used the a priori equality (4.5), which only holds for a solution $w = S(t)u_0^\omega + v$ to (4.2).

In order to overcome this issue, we use an approximation method to construct a solution to (1.6). To be more precise, we construct a local solution u to (1.6) as a limit of a sequence $\{u^N\}_{N\in\mathbb{N}}$ of smooth solutions with smooth initial data $u_{0,N}^\omega$. For simplicity of the presentation, we only consider the following frequency-truncated data:

$$u_{0,N}^\omega := \pi_N u_0^\omega = \sum_{|n|\le N} g_n(\omega)e^{inx}$$

in the following. Here, π_N is the Dirichlet frequency projection onto the frequencies $\{|n| \le N\}$ defined in (2.4). See Remark 4.4 (ii) for the case of smooth initial data given by mollification as in (1.4).

Letting

$$g_n^N := \mathbf{1}_{|n|\le N}\cdot g_n = \begin{cases} g_n, & \text{if } |n|\le N,\\ 0, & \text{if } |n| > N,\end{cases} \tag{4.8}$$

we have

$$u_{0,N}^\omega(x) = \sum_{n\in\mathbb{Z}} g_n^N(\omega)e^{inx}.$$

Define a truncated version of the random phase function Ψ^ω in (1.46) by setting

$$\Psi_N^\omega := |g_{n_1}^N(\omega)|^2 - |g_{n_2}^N(\omega)|^2 + |g_{n_3}^N(\omega)|^2 - |g_n^N(\omega)|^2. \tag{4.9}$$

We also set $\Psi_\infty^\omega = \Psi^\omega$.

Let $N\in\mathbb{N}$. Then, we have $u_{0,N}^\omega \in C^\infty(\mathbb{T})$ almost surely. Hence, by Proposition 1.1 in [57], there exists a unique global-in-time solution u^N to (1.6) with $u^N|_{t=0} = u_{0,N}^\omega$. Furthermore, by introducing the truncated random gauge transform:

$$w^N(x,t) = \mathcal{J}_N^\omega(u^N) := \sum_{n\in\mathbb{Z}} e^{inx - it|g_n^N(\omega)|^2}\widehat{u^N}(n,t) \tag{4.10}$$

with g_n^N in (4.8), we see that w^N satisfies a modified version of the random equation (1.44):

$$\begin{cases} i\partial_t w^N = \partial_x^4 w^N + \mathcal{N}_{1,N}^\omega(w^N) + \mathcal{N}_{2,N}^\omega(w^N) \\ w|_{t=0} = u_{0,N}^\omega, \end{cases} \tag{4.11}$$

where $\mathcal{N}_{1,N}^\omega(w) = \mathcal{N}_{1,N}^\omega(w,w,w)$ and $\mathcal{N}_{2,N}^\omega(w)$ are defined by

$$\mathcal{N}_{1,N}^\omega(w_1,w_2,w_3)(x,t) := \sum_{n\in\mathbb{Z}} e^{inx} \sum_{\Gamma(n)} e^{it\Psi_N^\omega(\bar{n})} \widehat{w}_1(n_1,t)\overline{\widehat{w}_2(n_2,t)}\widehat{w}_3(n_3,t), \tag{4.12}$$

$$\mathcal{N}_{2,N}^\omega(w)(x,t) := -\sum_{n\in\mathbb{Z}} e^{inx}\big[|\widehat{w}(n,t)|^2 - |g_n^N(\omega)|^2\big]\widehat{w}(n,t).$$

By writing (4.11) in the Duhamel formulation, we have

$$w^N = S(t)u_{0,N}^\omega + \mathcal{I}_{1,N}^\omega(w^N) + \mathcal{I}_{2,N}^\omega(w^N), \tag{4.13}$$

where $\mathcal{I}_{1,N}^\omega(w) := \mathcal{I}_{1,N}^\omega(w,w,w)$ and $\mathcal{I}_{2,N}^\omega(w)$ are defined by

$$\mathcal{I}_{1,N}^\omega(w_1,w_2,w_3) := -i\int_0^t S(t-t')\mathcal{N}_{1,N}^\omega(w_1,w_2,w_3)(t')dt', \tag{4.14}$$

$$\mathcal{I}_{2,N}^\omega(w) := -i\int_0^t S(t-t')\mathcal{N}_{2,N}^\omega(w)(t')dt'. \tag{4.15}$$

Noting that w^N is almost surely a smooth solution to (4.11) with the truncated random initial data $u_{0,N}^\omega$, we have

$$\begin{aligned} |\widehat{w^N}(n,t)|^2 - |g_n^N|^2 &= \int_0^t \frac{d}{dt}|\widehat{w^N}(n,t')|^2 dt' \\ &= -2\operatorname{Re} i\int_0^t \sum_{\Gamma(n)} e^{it'\Psi_N^\omega(\bar{n})}\widehat{w^N}(n_1,t')\overline{\widehat{w^N}(n_2,t')}\widehat{w^N}(n_3,t')\overline{\widehat{w^N}(n,t')}dt' \\ &=: \mathcal{E}_n^N(w^N,w^N,w^N,w^N)(t). \end{aligned} \tag{4.16}$$

This motivates us to define a truncated version of $\widetilde{\mathcal{I}}_2$ by

$$\begin{aligned} &\widetilde{\mathcal{I}}_{2,N}^\omega(w_1,w_2,w_3,w_4,w_5)(x,t) \\ &\qquad := \int_0^t S(t-t')\sum_{n\in\mathbb{Z}} e^{inx}\mathcal{E}_n^N(w_1,w_2,w_3,w_4)(t')\widehat{w}_5(n,t')dt'. \end{aligned} \tag{4.17}$$

We also set $\widetilde{\mathcal{I}}_{2,N}^\omega(w^N) = \widetilde{\mathcal{I}}_{2,N}^\omega(w^N,w^N,w^N,w^N,w^N)$. Then, we can rewrite (4.13) as the following partially iterated Duhamel formulation:

$$w^N = S(t)u_{0,N}^\omega + \mathcal{I}_{1,N}^\omega(w^N) + \widetilde{\mathcal{I}}_{2,N}^\omega(w^N). \tag{4.18}$$

Note that while $\mathcal{I}_{2,N}^\omega(w)$ in (4.15) corresponds to the resonant part of the nonlinearity, only the non-resonant contribution survives in (4.16) after substituting the equation, thus yielding a non-resonant structure in the quintic term $\widetilde{\mathcal{I}}_{2,N}^\omega(w^N)$.

In order to prove Theorem 2, we need to show that $\{w^N\}_{N\in\mathbb{N}}$ converges in some function space and that the limit $w = \lim_{N\to\infty} w^N$ is a distributional solution to (1.44). We now state the crucial nonlinear estimates in our analysis. Recall from (2.5) that given $N \in \mathbb{Z}_{\geq -1} = \mathbb{Z}\cap[-1,\infty)$, $\pi_N^\perp$ denotes the frequency projection operator onto the (spatial) frequencies $\{|n| > N\}$ with the understanding that $\pi_{-1}^\perp = \operatorname{Id}$.

Proposition 4.1. *Let $0 < \beta, \gamma \ll 1$ and $b > \frac{1}{2}$ be sufficiently close to $\frac{1}{2}$. Then, there exist $c, \theta > 0$ and small $\delta_0 > 0$ with the following property. For each $0 < \delta < \delta_0$, there exists $\Omega_\delta \subset \Omega$ with $P(\Omega_\delta^c) < e^{-\frac{1}{\delta^c}}$ such that for each $\omega \in \Omega_\delta$, we have*

$$\|\mathcal{I}^\omega_{1,N}(w_1, w_2, w_3)\|_{X^{0,b,\delta}} \leq C\delta^\theta \prod_{j=1}^3 \Big(\langle N_j\rangle^{-\beta} + \|w_j - S(t)\pi^\perp_{N_j}(u_0^\omega)\|_{X^{-\gamma,b,\delta}}\Big), \tag{4.19}$$

uniformly in $N_j \in \mathbb{Z}_{\geq -1}$, $j = 1, 2, 3$, and $N \geq N_0(\omega, \delta)$ for some $N_0(\omega, \delta) \in \mathbb{N}$. Here, we allow $N = \infty$ as well.

Proposition 4.2. *Let $0 < \beta, \gamma \ll 1$ and $b > \frac{1}{2}$ be sufficiently close to $\frac{1}{2}$. Then, there exist $c, \theta > 0$ and small $\delta_0 > 0$ with the following property. For each $0 < \delta < \delta_0$, there exists $\Omega_\delta \subset \Omega$ with $P(\Omega_\delta^c) < e^{-\frac{1}{\delta^c}}$ such that for each $\omega \in \Omega_\delta$, we have*

$$\begin{aligned}\|\widetilde{\mathcal{I}}^\omega_{2,N}&(w_1, w_2, w_3, w_4, w_5)\|_{X^{0,b,\delta}}\\ &\leq C\delta^\theta \prod_{j=1}^5 \Big(\langle N_j\rangle^{-\beta} + \|w_j - S(t)\pi^\perp_{N_j}(u_0^\omega)\|_{X^{-\gamma,b,\delta}}\Big),\end{aligned} \tag{4.20}$$

uniformly in $N_j \in \mathbb{Z}_{\geq -1}$, $j = 1, \dots, 5$, and $N \geq N_0(\omega, \delta)$ for some $N_0(\omega, \delta) \in \mathbb{N}$. Here, we allow $N = \infty$ as well.

We remark that both estimates (4.19) and (4.20) exhibit some smoothing effect. The main reason is that both nonlinearities $\mathcal{I}^\omega_{1,N}(w_1, w_2, w_3)$ and $\widetilde{\mathcal{I}}^\omega_{2,N}(w_1, w_2, w_3, w_4, w_5)$ possess non-resonant structures. In the next subsection, we present the proof of Theorem 2 by assuming Propositions 4.1 and 4.2. We present the proofs of these propositions in Sections 6 and 7. By careful analysis, we reduce these nonlinear estimates to boundedness properties of certain random multilinear functionals of the white noise.

Remark 4.3. In deriving $\mathcal{E}_n(w, w, w, w)$ in (4.4), we made use of a key cancellation:

$$\operatorname{Re}\Big(i\mathcal{F}\big(\mathcal{N}_2^\omega(w)\big)(n)\overline{\widehat{w}(n)}\Big) = 0, \tag{4.21}$$

i.e. the resonant part of the nonlinearity disappears in (4.4). Interestingly, a similar cancellation is used in the context of the modified scattering analysis of the one-dimensional cubic nonlinear Schrödinger equation on the real line:

$$i\partial_t u = \partial_x^2 u + |u|^2 u \tag{4.22}$$

with localized initial data. More precisely, if we set $v(t) = e^{it\partial_x^2}u(t)$, then by a stationary phase argument, (4.22) can be rewritten as

$$\partial_t \widehat{v}(\xi, t) = cit^{-1}|\widehat{v}(\xi, t)|^2\widehat{v}(\xi, t) + R(\xi, t), \qquad \xi \in \mathbb{R}, \tag{4.23}$$

where c is a real constant and $\widehat{v}$ denotes the Fourier transform of v on the real line. The trilinear remainder term $R(\xi, t)$ decays (in a suitable functional framework) faster than t^{-1} and therefore the principal part of the nonlinearity for analyzing long-time behavior is given by $cit^{-1}|\widehat{v}(\xi, t)|^2\widehat{v}(\xi, t)$, which is the analogue of the resonant part of the nonlinearity $\mathcal{N}_2^\omega(w)$ in our problem. Note that the key cancellation in the context of (4.23) is

$$\operatorname{Re}\Big(it^{-1}|\widehat{v}(\xi, t)|^2\widehat{v}(\xi, t)\overline{\widehat{v}(\xi, t)}\Big) = 0. \tag{4.24}$$

The cancellation (4.24) appears in computing $\partial_t|\widehat{v}(\xi, t)|^2$, which is the analogue of the computation (4.4) in the context of (4.23). We point out strong similarity between (4.21) and (4.24).

4.2. **Proof of Theorem 2: the $\alpha = 0$ case.** In this subsection, we present the proof of Theorem 2 for $\alpha = 0$. More precisely, by applying Propositions 4.1 and 4.2 to the iterated Duhamel formulation (4.18) we prove that, for each $0 < \delta \ll 1$, there exists $\Omega_\delta \subset \Omega$ with $P(\Omega_\delta^c) \le e^{-\frac{1}{\delta^c}}$ such that for $\omega \in \Omega_\delta$, the following statements hold:

(i) The sequence $\{w^N - S(t)u_{0,N}^\omega\}_{N\in\mathbb{N}}$ is Cauchy in $X^{0,\frac{1}{2}+,\delta}$.

(ii) The limit w of w^N satisfies the equation (1.44) in the distributional sense with the white noise initial data u_0^ω.

(iii) The solution w is unique in the class: $S(t)u_0^\omega + B_1$, where B_1 denotes the ball of radius 1 in $X^{0,\frac{1}{2}+,\delta}$ centered at the origin.

Given $0 < \beta, \gamma \ll 1$ and $b > \frac{1}{2}$ sufficiently close to $\frac{1}{2}$, apply Propositions 4.1 and 4.2 and construct a set $\Omega_\delta \subset \Omega$ with $P(\Omega_\delta^c) < e^{-\frac{1}{\delta^c}}$ for each $0 < \delta \ll 1$ such that the conclusions of both Propositions 4.1 and 4.2 hold. In the following, we fix $\omega \in \Omega_\delta$ and hence the parameter $N_0(\omega, \delta)$ in Propositions 4.1 and 4.2 is a *fixed* number. In what follows, unless otherwise stated, the number N and M are always assumed to be greater than $N_0(\omega, \delta)$.

(i) By setting $v^N = w^N - S(t)u_{0,N}^\omega$, it follows from (4.13) and (4.18) that v^N satisfies

$$\begin{aligned} v^N &= \mathcal{I}_{1,N}^\omega(v^N + S(t)u_{0,N}^\omega) + \mathcal{I}_{2,N}^\omega(v^N + S(t)u_{0,N}^\omega) \\ &= \mathcal{I}_{1,N}^\omega(v^N + S(t)u_{0,N}^\omega) + \widetilde{\mathcal{I}}_{2,N}^\omega(v^N + S(t)u_{0,N}^\omega), \end{aligned} \tag{4.25}$$

where $\mathcal{I}_{1,N}^\omega$, $\mathcal{I}_{2,N}^\omega$ and $\widetilde{\mathcal{I}}_{2,N}^\omega$ are as in (4.14), (4.15) and (4.17). Note that the second equality holds since w^N is a classical solution to (4.11).

We first claim that

$$\|v^N\|_{X^{0,\frac{1}{2}+,\delta}} \le 1 \tag{4.26}$$

by choosing $\delta > 0$ sufficiently small. Indeed, by applying (4.19) and (4.20) in Propositions 4.1 and 4.2 (with $N_j = -1$, i.e. $\pi_{N_j}^\perp = \mathrm{Id}$) to (4.25), we have

$$\begin{aligned} \|v^N\|_{X^{0,\frac{1}{2}+,\delta}} &\lesssim \delta^\theta(1 + \|v^N\|_{X^{-\gamma,\frac{1}{2}+,\delta}})^3 + \delta^\theta(1 + \|v^N\|_{X^{-\gamma,\frac{1}{2}+,\delta}})^5 \\ &\le \delta^\theta(1 + \|v^N\|_{X^{0,\frac{1}{2}+,\delta}})^3 + \delta^\theta(1 + \|v^N\|_{X^{0,\frac{1}{2}+,\delta}})^5. \end{aligned} \tag{4.27}$$

Then by choosing $\delta > 0$ sufficiently small, the bound (4.26) follows from (4.27) and a standard continuity argument.

Next, we show that the sequence $\{v^N\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $X^{0,\frac{1}{2}+,\delta}$. By possibly restricting to smaller $\delta > 0$, we prove

$$\|v^M - v^N\|_{X^{0,\frac{1}{2}+,\delta}} \lesssim N^{-\min(\beta,\gamma)} \tag{4.28}$$

for any $\omega \in \Omega_\delta$ and $M \ge N \ge N_0(\omega, \delta)$. The bound (4.28) shows that v^N converge in $X^{0,\frac{1}{2}+,\delta}$ for each $\omega \in \Omega_\delta$ and thus $w^N = v^N + S(t)u_{0,N}^\omega$ converge to $w = v + S(t)u_0^\omega$ in $C([-\delta,\delta]; H^s(\mathbb{T}))$, $s < -\frac{1}{2}$.

We now prove (4.28). From (4.25), we have

$$\begin{aligned} \|v^M - v^N\|_{X^{0,\frac{1}{2}+,\delta}} &\le \|\mathcal{I}_{1,M}^\omega(v^M + S(t)u_{0,M}^\omega) - \mathcal{I}_{1,N}^\omega(v^N + S(t)u_{0,N}^\omega)\|_{X^{0,\frac{1}{2}+,\delta}} \\ &\quad + \|\widetilde{\mathcal{I}}_{2,M}^\omega(v^M + S(t)u_{0,M}^\omega) - \widetilde{\mathcal{I}}_{2,N}^\omega(v^N + S(t)u_{0,N}^\omega)\|_{X^{0,\frac{1}{2}+,\delta}}. \end{aligned} \tag{4.29}$$

We first estimate the first term on the right-hand side of (4.29). From (4.12) and (4.14) with $w^N = v^N + S(t)u^\omega_{0,N}$, we have

$$\begin{aligned}
&\|\mathcal{I}^\omega_{1,M}(w^M) - \mathcal{I}^\omega_{1,N}(w^N)\|_{X^{0,\frac12+,\delta}} \\
&\le \|\mathcal{I}^\omega_{1,M}(w^M) - \mathcal{I}^\omega_{1,N}(w^M)\|_{X^{0,\frac12+,\delta}} + \|\mathcal{I}^\omega_{1,N}(w^M) - \mathcal{I}^\omega_{1,N}(w^N)\|_{X^{0,\frac12+,\delta}} \\
&\le \|\mathcal{I}^\omega_{1,M}(w^M) - \mathcal{I}^\omega_{1,N}(w^M)\|_{X^{0,\frac12+,\delta}} + \|\mathcal{I}^\omega_{1,N}(w^M - w^N, w^M, w^M)\|_{X^{0,\frac12+,\delta}} \\
&\quad + \|\mathcal{I}^\omega_{1,N}(w^N, w^M - w^N, w^M)\|_{X^{0,\frac12+,\delta}} + \|\mathcal{I}^\omega_{1,N}(w^N, w^N, w^M - w^N)\|_{X^{0,\frac12+,\delta}}.
\end{aligned} \tag{4.30}$$

In the following, we only treat the first two terms since the other two terms can be treated in a similar manner. Using the trilinear structure of $\mathcal{I}^\omega_{1,L}$ for $L \in \{M, N\}$, we have

$$\begin{aligned}
\mathcal{I}^\omega_{1,L}(w^M) &= \mathcal{I}^\omega_{1,L}(\pi^\perp_{\frac N3} w^M, w^M, w^M) + \mathcal{I}^\omega_{1,L}(\pi_{\frac N3} w^M, \pi^\perp_{\frac N3} w^M, w^M) \\
&\quad + \mathcal{I}^\omega_{1,L}(\pi_{\frac N3} w^M, \pi_{\frac N3} w^M, \pi^\perp_{\frac N3} w^M) + \mathcal{I}^\omega_{1,L}(\pi_{\frac N3} w^M, \pi_{\frac N3} w^M, \pi_{\frac N3} w^M).
\end{aligned}$$

The key point is to observe that it follows directly from the definitions (4.12) and (4.14) with (4.9) that for $M \ge N$

$$\mathcal{I}^\omega_{1,M}(\pi_{\frac N3} w^M, \pi_{\frac N3} w^M, \pi_{\frac N3} w^M) - \mathcal{I}^\omega_{1,N}(\pi_{\frac N3} w^M, \pi_{\frac N3} w^M, \pi_{\frac N3} w^M) = 0.$$

Therefore, in order to control

$$\|\mathcal{I}^\omega_{1,M}(w^M) - \mathcal{I}^\omega_{1,N}(w^M)\|_{X^{0,\frac12+,\delta}},$$

we only need to bound

$$\begin{aligned}
&\|\mathcal{I}^\omega_{1,L}(\pi^\perp_{\frac N3} w^M, w^M, w^M)\|_{X^{0,\frac12+,\delta}}, \\
&\|\mathcal{I}^\omega_{1,L}(\pi_{\frac N3} w^M, \pi^\perp_{\frac N3} w^M, w^M)\|_{X^{0,\frac12+,\delta}}, \\
&\|\mathcal{I}^\omega_{1,L}(\pi_{\frac N3} w^M, \pi_{\frac N3} w^M, \pi^\perp_{\frac N3} w^M)\|_{X^{0,\frac12+,\delta}}
\end{aligned}$$

for $L = M$ and N. We only consider the first one since the others can be treated similarly. From Proposition 4.1 and (4.26), we have

$$\begin{aligned}
\|\mathcal{I}^\omega_{1,L}(\pi^\perp_{\frac N3} w^M, w^M, w^M)\|_{X^{0,\frac12+,\delta}} &\lesssim \delta^\theta \big(N^{-\beta} + \|\pi^\perp_{\frac N3} v^M\|_{X^{-\gamma,\frac12+,\delta}}\big)\big(1 + \|v^M\|_{X^{-\gamma,\frac12+,\delta}}\big)^2 \\
&\lesssim \delta^\theta \big(N^{-\beta} + N^{-\gamma}\|v^M\|_{X^{0,\frac12+,\delta}}\big) \\
&\lesssim \delta^\theta N^{-\min(\beta,\gamma)},
\end{aligned}$$

where we used the fact that $w^N = v^N + S(t)u^\omega_{0,N}$. Therefore, we obtain

$$\|\mathcal{I}^\omega_{1,M}(w^M) - \mathcal{I}^\omega_{1,N}(w^M)\|_{X^{0,\frac12+,\delta}} \lesssim \delta^\theta N^{-\min(\beta,\gamma)}.$$

Next, we proceed with estimating the second term on the right-hand side of (4.30):

$$\begin{aligned}
&\|\mathcal{I}^\omega_{1,N}(w^M - w^N, w^M, w^M)\|_{X^{0,\frac12+,\delta}} \\
&\quad \le \|\mathcal{I}^\omega_{1,N}(v^M - v^N + S(t)\pi^\perp_N u^\omega_0, v_N + S(t)u^\omega_{0,N}, v_N + S(t)u^\omega_{0,N})\|_{X^{0,\frac12+,\delta}} \\
&\qquad + \|\mathcal{I}^\omega_{1,N}(S(t)\pi^\perp_M u^\omega_0, v_N + S(t)u^\omega_{0,N}, v_N + S(t)u^\omega_{0,N})\|_{X^{0,\frac12+,\delta}}.
\end{aligned} \tag{4.31}$$

By applying Proposition 4.1 to (4.31) with $N_1 = N$ or M and $N_2 = N_3 = -1$, we obtain

$$\begin{aligned}
&\|\mathcal{I}^\omega_{1,N}(w^M - w^N, w^M, w^M)\|_{X^{0,\frac12+,\delta}} \\
&\quad\lesssim \delta^\theta\Big(N^{-\beta} + \|v^M - v^N\|_{X^{-\gamma,\frac12+,\delta}}\Big)\Big(1 + \|v^M\|_{X^{-\gamma,\frac12+,\delta}}\Big)^2 \\
&\quad\lesssim \delta^\theta\Big(N^{-\beta} + \|v^M - v^N\|_{X^{-\gamma,\frac12+,\delta}}\Big).
\end{aligned} \tag{4.32}$$

Similarly, we can estimate the second term on the right-hand side of (4.29) by applying Proposition 4.2 and obtain

$$\begin{aligned}
&\|\widetilde{\mathcal{I}}^\omega_{2,M}(w^M) - \widetilde{\mathcal{I}}^\omega_{2,N}(w^N)\|_{X^{0,\frac12+,\delta}} \\
&\qquad\lesssim \delta^\theta\Big(N^{-\min(\beta,\gamma)} + \|v^M - v^N\|_{X^{0,\frac12+,\delta}}\Big).
\end{aligned} \tag{4.33}$$

Putting (4.29), (4.32), and (4.33) together, we obtain

$$\|v^M - v^N\|_{X^{0,\frac12+,\delta}} \le C\delta^\theta N^{-\min(\beta,\gamma)} + C\delta^\theta\|v^N - v^M\|_{X^{0,\frac12+,\delta}}.$$

Therefore, by choosing $\delta > 0$ sufficiently small, we obtain (4.28).

(ii) Next, we show that the limit $w = v + S(t)u_0^\omega$ satisfies the Duhamel formulation (4.2):

$$w = S(t)u_0^\omega + \mathcal{I}_1(w) + \mathcal{I}_2(w), \tag{4.34}$$

in the distributional sense, locally in time.

Given $0 < \delta \ll 1$, let $\omega \in \Omega_\delta$. Then, it follows from Lemma 2.7[11] that the truncated random linear solution $S(t)u^\omega_{0,N}$ converges to $S(t)u_0^\omega$ in $C([-\delta,\delta];\mathcal{F}L^{-\varepsilon,\infty}(\mathbb{T}))$ for any $\varepsilon > 0$. The residual part v^N converges to v in $X^{0,\frac12+,\delta}$, and hence in $C([-\delta,\delta];L^2(\mathbb{T}))$. Putting together, we see that w^N converges to w in $C([-\delta,\delta];\mathcal{F}L^{-\varepsilon,\infty}(\mathbb{T}))$. Hence, from the definitions (4.1) and (4.15) of $\mathcal{I}_2$ and $\mathcal{I}_{2,N}$, we conclude that $\mathcal{I}_{2,N}(w^N)$ converges to $\mathcal{I}_2(w)$ in $C([-\delta,\delta];\mathcal{F}L^{-3\varepsilon,\infty}(\mathbb{T}))$. On the other hand, from (4.30), we see that that $\mathcal{I}_{1,N}(w^N)$ converges to $\mathcal{I}_1(w)$ in $X^{0,\frac12+,\delta}$. Together with the convergence of w^N to w, we have shown that each term in the truncated Duhamel formulation (4.13) converges to the corresponding term in (4.34). Recalling that w^N satisfies (4.13), we conclude that w is a solution to the Duhamel formulation (4.34) in the distributional sense.

In Step (i), we already showed that w satisfies the iterated formulation (4.5). Thus, as a byproduct, we have verified that

$$\mathcal{I}_2(w) = \widetilde{\mathcal{I}}_2(w),$$

for the solution w constructed in Step (i).

(iii) Lastly, we turn to the uniqueness issue. Given $0 < \delta \ll 1$, fix $\omega \in \Omega_\delta$. Let $w = S(t)u_0^\omega + v$ be the solution to (4.2) with the white noise initial data u_0^ω constructed in Steps (i) and (ii). Suppose that there exists another solution $\widetilde{w}$ to (4.2) of the form $\widetilde{w} = S(t)u_0^\omega + \widetilde{v}$ for some $\widetilde{v} \in B_1 \subset X^{0,\frac12+,\delta}$. Since such $\widetilde{w}$ is also a solution to (1.44), by repeating the argument in Subsection 4.1, we see that $\widetilde{w}$ satisfies the iterated Duhamel formulation (4.6):

$$\widetilde{w} = S(t)u_0^\omega + \mathcal{I}_1(\widetilde{w}) + \widetilde{\mathcal{I}}_2(\widetilde{w}).$$

[11]Note that Lemma 2.7 appears in the proof of Propositions 4.1 and 4.2 (see also Lemma A.3) and thus we may assume that the conclusion of Lemma 2.7 holds on the set Ω_δ constructed in Step (i).

Then, by repeating the argument in Step (i) with Propositions 4.1 and 4.2, we obtain

$$\|v-\widetilde{v}\|_{X^{0,\frac{1}{2}+,\delta}} \le C\delta^\theta \|v-\widetilde{v}\|_{X^{0,\frac{1}{2}+,\delta}} \le \frac{1}{2}\|v-\widetilde{v}\|_{X^{0,\frac{1}{2}+,\delta}}$$

for $\delta > 0$ sufficiently small, yielding $v = \widetilde{v}$ in $X^{0,\frac{1}{2}+,\delta}$. This proves uniqueness in the class $S(t)u_0^\omega + B_1$.

This completes the proof of Theorem 2 when $\alpha = 0$.

Remark 4.4. (i) By a continuity argument, we can easily upgrade the uniqueness of w in $S(t)u_0^\omega + B_1$ to uniqueness of w in the class

$$S(t)u_0^\omega + X^{0,\frac{1}{2}+,\delta}$$

See Remark 2.9 in [20]. By inverting the random gauge transform $\mathcal{J}^\omega$ in (1.42), we then obtain uniqueness of u in the class

$$Z(u_0^\omega) + X^{0,\frac{1}{2}+,\delta}_{-,\omega}$$

where Z is as in (1.11) and $X^{0,\frac{1}{2}+,\delta}_{-,\omega}$ is the local-in-time version of the *random* Fourier restriction norm space $X^{0,\frac{1}{2}+}_{-,\omega}$ defined in (A.2).

(ii) Let $u_{0,m}^\omega = u_0^\omega * \rho_m$ be the regularization of the white noise u_0^ω by mollification via a mollification kernel ρ_m in (1.4). Denote by w_m the solution to the gauged equation (1.44) with $w_m|_{t=0} = u_{0,m}^\omega$. Then, by proceeding as above,[12] one can easily establish convergence of w_m to $\widetilde{w}$ in the class $S(t)u_0^\omega + B_1$, satisfying (4.2). Then, by the uniqueness proved in Step (iii) above, we conclude that $w = \widetilde{w}$. This proves independence of the mollification kernel.

5. Global well-posedness and invariance of the white noise measure

In this section, we extend the local solutions constructed in Theorem 2 to global solutions and prove invariance of the white noise measure (1.7) with $\alpha = 0$ under the flow of the renormalized 4NLS (1.6). The main ingredient is Bourgain's invariant measure argument [6, 7].

5.1. Invariance of the white noise measure under the truncated 4NLS.

In this section, we will denote the white noise measure by μ. For fixed $\varepsilon > 0$, μ is a measure on $H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$, defined as the pushforward of P under the map from $(\Omega, \mathcal{F}, P)$ to $H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$ (equipped with the Borel σ-algebra) given by

$$\omega \longmapsto u_0^\omega = \sum_{n\in\mathbb{Z}} g_n(\omega)e^{inx}.$$

Given $N \in \mathbb{N}$, we also define the finite-dimensional white noise measure μ_N on $E_N = \operatorname{span}\{e^{inx}, |n| \le N\}$ as the pushforward of P under the map from $(\Omega, \mathcal{F}, P)$ to E_N given by $\omega \mapsto \pi_N u_0^\omega$, where π_N is the Dirichlet projector onto the frequencies $\{|n| \le N\}$ defined in (2.4).

Consider the frequency-truncated version of the renormalized 4NLS (1.6):

$$\begin{cases} i\partial_t u^N = \partial_x^4 u^N + \pi_N(\mathcal{N}(u^N)) \\ u^N(x,0) = \pi_N u_0(x) \in E_N, \end{cases} \tag{5.1}$$

where $\mathcal{N}(u)$ denotes the renormalized nonlinearity in (1.17). It is easy to see that the solution u^N to (5.1) exists globally in time. Let $\widetilde{\Theta}_N(t)$ denote the flow map for (5.1). By the Liouville theorem, we see that the truncated white noise measure μ_N is invariant under $\widetilde{\Theta}_N(t)$. Following

[12] Here, our assumption that the symbol $\widehat{\rho}_m \equiv 1$ on $[-c_0 m, c_0 m]$ for some $c_0 > 0$, independent of $m \in \mathbb{N}$ provides a simplification of the argument as compared to a general mollification kernel.

[14], we also consider the extension of (5.1) to infinite dimensions, where the higher modes evolve according to linear dynamics:

$$\begin{cases} i\partial_t u^N = \partial_x^4 u^N + \pi_N(\mathcal{N}(\pi_N u^N)) \\ u^N(x,0) = u_0(x) \in H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}). \end{cases} \tag{5.2}$$

Let $\Theta_N(t)$ denote the flow map for (5.2). Then, we have

$$\Theta^N(t) = \widetilde{\Theta}^N(t)\pi_N + S(t)\pi_N^\perp,$$

where $\pi_N^\perp = \mathrm{Id} - \pi_N$. Denoting by $E_N^\perp$ the orthogonal complement of E_N in $H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$, let $\mu_N^\perp$ be the white noise measure on $E_N^\perp$ (i.e. the image measure under the map: $\omega \mapsto \pi_N^\perp u_0^\omega$). Note that $\mu_N^\perp$ is invariant along the linear flow on $E_N^\perp$ (this is a consequence of the invariance of complex-valued Gaussians under rotations). Therefore, by writing

$$d\mu = d\mu_N \otimes d\mu_N^\perp,$$

we conclude the following invariance of μ under $\Theta_N(t)$.

Lemma 5.1. *For each $t \in \mathbb{R}$, the white noise measure μ is invariant under the flow map $\Theta^N(t)$ on $H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$.*

5.2. **Almost sure global well-posedness.** By using the invariance of the white noise measure for (5.2) (Lemma 5.1) and a PDE approximation argument, we have the following lemma, guaranteeing long time existence with large probability for the renormalized 4NLS (1.6).

Lemma 5.2. *There exist small $0 < \varepsilon < \varepsilon_1 \ll 1$ and $\beta > 0$ such that given any small $\kappa > 0$ and $T > 0$, there exists a measurable set $\Sigma_{\kappa,T} \subset H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$ such that* (i) *$\mu(\Sigma_{\kappa,T}^c) < \kappa$ and* (ii) *for any $u_0 \in \Sigma_{\kappa,T}$, there exists a (unique) solution*

$$u \in Z(u_0) + C([-T,T]; L^2(\mathbb{T})) \subset C([-T,T]; H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}))$$

to the renormalized 4NLS (1.6) *with $u|_{t=0} = u_0$, where Z is defined in* (1.11)*. Furthermore, given any large $N \gg 1$, we have*

$$\Big\| u(t) - \Theta^N(t)(u_0) \Big\|_{C([-T,T]:H^{-\frac{1}{2}-\varepsilon_1}(\mathbb{T}))} \lesssim C(\kappa, T) N^{-\beta},$$

where $\Theta^N(t)$ denotes the flow map for (5.2).

For the uniqueness statement, see Remark 4.4 (i).

Proof. Once we have almost sure local well-posedness (Theorem 2), the proof of Lemma 5.2 is by now standard. In the following, we only sketch key parts of the argument and refer to [6, 7, 16, 64, 65] for further details.

Given a solution u^N to (5.2), we define $w^N = \mathcal{J}_N^\omega(u^N)$ as in the proof of Theorem 2, where $\mathcal{J}_N^\omega$ denotes the truncated random gauge transform in (4.10). Namely, we have

$$w^N(x,t) = \sum_{n \in \mathbb{Z}} e^{inx - it|g_n^N(\omega)|^2} \widehat{u^N}(n,t),$$

where g_n^N is as in (4.8). The key observation is that convergence properties of w^N in a Fourier lattice[13] can be directly converted to convergence properties of u^N. For $M > N \geq 1$, write

$$w^M - w^N = \big(\pi_M w^M - \pi_N w^N\big) + \pi_M^\perp w^M - \pi_N^\perp w^N.$$

[13]Namely, in a space where a norm depends only on the sizes of the Fourier coefficients. For example, $H^s(\mathbb{T})$ and $\mathcal{F}L^{s,p}(\mathbb{T})$.

The convergence of $(\pi_M w^M - S(t)u^\omega_{0,M}) - (\pi_N w^N - S(t)u^\omega_{0,N})$ can be shown exactly as in the proof of Theorem 2, locally in time, i.e. in $X^{0,\frac{1}{2}+,\delta} \subset C([-\delta,\delta];L^2(\mathbb{T}))$, which yields convergence of $\pi_M w^M - \pi_N w^N$ in $C([-\delta,\delta];H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}))$. On the other hand, the second and third terms decay like $N^{-\beta}$ for some $\beta > 0$ thanks to the high frequency projections. The remaining part of the argument leading to the proof of Lemma 5.2 is contained in [6, 7, 16, 64, 65]. In particular, see the proof of Proposition 3.5 in [64] for details in a setting analogous to our work. □

Once we have Lemma 5.2, the desired almost sure global well-posedness follows from the Borel-Cantelli lemma. Given $\kappa > 0$, let $T_j = 2^j$ and $\kappa_j = \frac{\kappa}{2^j}$, $j \in \mathbb{N}$. By applying Lemma 5.2, construct a set Σ_{κ_j,T_j} and set

$$\Sigma_\kappa := \bigcap_{j=1}^\infty \Sigma_{\kappa_j,T_j}. \tag{5.3}$$

Then, we have $\mu(\Sigma_\kappa^c) < \kappa$ and for any $u_0 \in \Sigma_\kappa$, there exists a unique global-in-time solution to the renormalized 4NLS (1.6) with $u|_{t=0} = u_0$. Finally, set

$$\Sigma := \bigcup_{n=1}^\infty \Sigma_{\frac{1}{n}}.$$

Then, we have $\mu(\Sigma^c) = 0$ and for any $u_0 \in \Sigma$, there exists a unique global-in-time solution to the renormalized 4NLS (1.6) with $u|_{t=0} = u_0$. This proves almost sure global well-posedness.

5.3. **Invariance of the white noise measure.** Let $\Theta(t)$ be the flow map for the renormalized 4NLS (1.6) defined on the set Σ of full probability constructed above. Our goal here is to show that

$$\int_\Sigma F\big(\Theta(t)(u)\big)d\mu(u) = \int_\Sigma F(u)d\mu(u) \tag{5.4}$$

for any $F \in L^1(H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}), d\mu)$ and any $t \in \mathbb{R}$. By a density argument, it suffices to prove (5.4) for continuous and bounded F.

Fix $t \in \mathbb{R}$. By Lemma 5.1, we have

$$\int_\Sigma F\big(\Theta^N(t)(u)\big)d\mu(u) = \int_\Sigma F(u)d\mu(u). \tag{5.5}$$

Fix small $\delta > 0$. The boundedness of F implies that for any sufficiently small $\kappa > 0$, we have

$$\bigg|\int_{\Sigma_\kappa^c} F\big(\Theta(t)(u)\big)d\mu(u)\bigg| + \bigg|\int_{\Sigma_\kappa^c} F\big(\Theta^N(t)(u)\big)d\mu(u)\bigg| < \delta, \tag{5.6}$$

where Σ_κ is as in (5.3). Fix one such $\kappa > 0$. Then, by Lemma 5.2, we have

$$\|\Theta(t)(u) - \Theta^N(t)(u)\|_{H^{-\frac{1}{2}-\varepsilon}} \le C(\kappa,t)N^{-\beta}$$

for any $u \in \Sigma_\kappa$ and sufficiently large $N \gg 1$. Hence, by continuity of F, we have

$$\bigg|\int_{\Sigma_\kappa} F\big(\Theta(t)(u)\big)d\mu(u) - \int_{\Sigma_\kappa} F\big(\Theta^N(t)(u)\big)d\mu(u)\bigg| < \delta, \tag{5.7}$$

for any sufficiently large $N \gg 1$. Combining (5.5), (5.6), and (5.7) and taking $\delta \to 0$, we obtain (5.4).

5.4. **Proof of Theorem 1.** The proof of Theorem 1 follows from the arguments presented in the proofs of Theorems 2 and 3.

6. Nonlinear estimate I: non-resonant part

In this section, we present the proof of Proposition 4.1.

6.1. **Probabilistic estimates.** We begin by presenting several probabilistic estimates that will be used to prove Proposition 4.1. The proofs of these lemmas are presented in Appendix A.

We first recall some notations. Let $\eta \in C_c^\infty(\mathbb{R})$ be a smooth non-negative cutoff function supported on $[-2,2]$ with $\eta \equiv 1$ on $[-1,1]$. Recall from (1.20), (2.6), and (4.9) that

$$\begin{aligned} \Gamma(n) &= \{(n_1,n_2,n_3) \in \mathbb{Z}^3 : n = n_1 - n_2 + n_3 \text{ and } n_1, n_3 \neq n\}, \\ \Phi(\bar n) &= \Phi(n_1,n_2,n_3,n) = n_1^4 - n_2^4 + n_3^4 - n^4, \\ \Psi_N^\omega(\bar n) &= |g_{n_1}^N(\omega)|^2 - |g_{n_2}^N(\omega)|^2 + |g_{n_3}^N(\omega)|^2 - |g_n^N(\omega)|^2. \end{aligned} \tag{6.1}$$

where g_n^N is as in (4.8). Given $s, b \in \mathbb{R}$ and $\delta > 0$, the following random functionals $S_{j,N}^{s,b,\delta}$, $j = 1,2,3$, play an important role in the proof of Proposition 4.1 (and also in the proof of Proposition 4.2 presented in Section 7):

$$S_{1,N}^{s,b,\delta}(f) = \Bigg\| \sum_{\substack{n_1 \in \mathbb{Z} \\ (n_1,n_2,n_3)\in\Gamma(n)}} \widehat f(n_1) \frac{\widehat\eta_\delta(\tau + \Phi(\bar n) - |g_{n_1}^N|^2)}{\langle n_2\rangle^s \langle n_3\rangle^s \langle n\rangle^{2s} \langle \tau\rangle^b} \Bigg\|_{\ell^2_{n,n_2,n_3} L^2_\tau} \tag{6.2}$$

(observe that there is at most one term in the n_1 summation),

$$\begin{aligned} S_{2,N}^{s,b,\delta}(f_1,f_2) = \Bigg\| & \sum_{\substack{n_1,n_2 \in \mathbb{Z} \\ (n_1,n_2,n_3)\in\Gamma(n)}} \widehat{f_1}(n_1)\overline{\widehat{f_2}(n_2)} \\ & \times \frac{\widehat\eta_\delta(\tau + \Phi(\bar n) - |g_{n_1}^N|^2 + |g_{n_2}^N|^2)}{\langle n_3\rangle^s \langle n\rangle^{2s} \langle \tau\rangle^b} \Bigg\|_{\ell^2_{n,n_3} L^2_\tau}, \end{aligned} \tag{6.3}$$

$$\begin{aligned} S_{3,N}^{s,b,\delta}(f_1,f_2,f_3) = \Bigg\| & \sum_{\Gamma(n)} \widehat{f_1}(n_1)\overline{\widehat{f_2}(n_2)}\widehat{f_3}(n_3) \\ & \times \frac{\widehat\eta_\delta(\tau + \Phi(\bar n) - |g_{n_1}^N|^2 + |g_{n_2}^N|^2 - |g_{n_3}^N|^2)}{\langle n\rangle^{2s} \langle \tau\rangle^b} \Bigg\|_{\ell^2_n L^2_\tau}. \end{aligned} \tag{6.4}$$

In the following, we will take f_1, f_2, f_3 as the white noise

$$f_1 = f_2 = f_3 = u_0^\omega = \sum_{n\in\mathbb{Z}} g_n(\omega) e^{inx}, \tag{6.5}$$

or its frequency truncated version (projected onto high frequencies)

$$\pi_{N_j}^\perp(u_0^\omega) = \sum_{|n|>N_j} g_n(\omega) e^{inx}.$$

For simplicity of notations, we set[14]

$$S_{1,N}^{s,b,\delta}(\omega) := S_{1,N}^{s,b,\delta}(\pi_{N_1}^{\perp}(u_0^\omega)), \tag{6.6}$$

$$S_{2,N}^{s,b,\delta}(\omega) := S_{2,N}^{s,b,\delta}(\pi_{N_1}^{\perp}(u_0^\omega), \pi_{N_2}^{\perp}(u_0^\omega)), \tag{6.7}$$

$$S_{3,N}^{s,b,\delta}(\omega) := S_{3,N}^{s,b,\delta}(\pi_{N_1}^{\perp}(u_0^\omega), \pi_{N_2}^{\perp}(u_0^\omega), \pi_{N_3}^{\perp}(u_0^\omega)), \tag{6.8}$$

for $N_1, N_2, N_3 \in \mathbb{Z}_{\geq -1}$ (recall our convention: $\pi_{-1}^{\perp} = \mathrm{Id}$). With the notations defined above, we have the following tail estimates for these random functionals.

Lemma 6.1. *Let $s < 0$, $b < \frac{1}{2}$, and $\beta > 0$ such that s and β are sufficiently close to 0 and b is sufficiently close to $\frac{1}{2}$. Then, there exist $c, \kappa > 0$ and small $\delta_0 > 0$ such that the following statements holds.*

(i) *We have*

$$P\bigg(\bigg\{\omega \in \Omega : \sup_{N \in \mathbb{N}} \sup_{N_1 \in \mathbb{Z}_{\geq -1}} \langle N_1 \rangle^\beta |S_{1,N}^{s,b,\delta}(\omega)| > \delta^\kappa\bigg\}\bigg) < e^{-\frac{1}{\delta^c}}$$

for any $0 < \delta < \delta_0$.

(ii) *Let $k = 2, 3$. Given $0 < \delta < \delta_0$, define the sets $\mathcal{A}_k$ by*

$$\mathcal{A}_k := \bigg\{\omega \in \Omega : \text{there exists } N_0 = N_0(\omega, \delta) \in \mathbb{N} \text{ such that}$$

$$\sup_{N \geq N_0} \sup_{\substack{N_j \in \mathbb{Z}_{\geq -1} \\ j = 1, \cdots, k}} \bigg(\prod_{j=1}^k \langle N_j \rangle^\beta\bigg) |S_{k,N}^{s,b,\delta}(\omega)| \leq \delta^\kappa\bigg\}.$$

Then, we have

$$P(\mathcal{A}_k^c) < e^{-\frac{1}{\delta^c}}$$

for any $0 < \delta < \delta_0$.

Given $N \in \mathbb{N} \cup \{\infty\}$, we introduce a random version $X_+^{s,b}(\omega, N)$ of the $X^{s,b}$-space:

$$\|u\|_{X_+^{s,b}(\omega,N)} = \|\langle n \rangle^s \langle \tau + n^4 + |g_n^N(\omega)|^2 \rangle^b \widehat{u}(n, \tau)\|_{\ell_n^2 L_\tau^2}$$

with the understanding that $g_n^\infty = g_n$. By slightly losing spatial regularity, we can control the random $X^{s,b}$-norm by the standard $X^{\sigma,b}$-norm (with $\sigma > s$) *uniformly* in $u \in X^{\sigma,b}$.

Lemma 6.2. *Let $\sigma > s$ and $b > 0$. Then, for each $K > 0$, there exists a set $\Omega_K \subset \Omega$ with $P(\Omega_K^c) < Ce^{-cK^{\frac{1}{b}}}$ such that*

$$\sup_{N \in \mathbb{N} \cup \{\infty\}} \|u\|_{X_+^{s,b}(\omega,N)} \lesssim (1 + K)\|u\|_{X^{\sigma,b}}$$

In particular, by choosing $K = \delta^{-\varepsilon}$ for some small $\varepsilon > 0$, there exists a set $\Omega_\delta \subset \Omega$ with $P(\Omega_K^c) < Ce^{-\frac{1}{\delta^c}}$ such that

$$\sup_{N \in \mathbb{N} \cup \{\infty\}} \|u\|_{X_+^{s,b}(\omega,N)} \lesssim \delta^{-\varepsilon}\|u\|_{X^{\sigma,b}}$$

uniformly in $u \in X^{\sigma,b}$, for any $0 < \delta \ll 1$.

[14] Strictly speaking, we should denote the dependence of $S_{j,N}^{s,b,\delta}(\omega)$ on the parameters N_1, N_2, and N_3. For simplicity of the presentation, however, we suppress such dependence unless it plays an important role.

For the proofs of Lemmas 6.1 and 6.2, see Appendix A. In the next subsection, we prove Proposition 4.1, assuming these lemmas.

6.2. **Proof of Proposition 4.1.** For $j = 1, 2, 3$, let $z_j = S(t)\pi^{\perp}_{N_j}(u_0^\omega)$ and set $v_j = w_j - z_j$. Then, by the linear estimate (Lemma 2.3), it suffices to construct $\Omega_\delta \subset \Omega$ with $P(\Omega_\delta^c) < e^{-\frac{1}{\delta^c}}$ such that for each $\omega \in \Omega_\delta$, we have, for some $s < 0$ sufficiently close to 0,

$$\|\mathcal{N}^\omega_{1,N}(v_1 + z_1, v_2 + z_2, v_3 + z_3)\|_{X^{0,-\frac{1}{2}+,\delta}} \le C\delta^\theta \prod_{j=1}^{3} \Big(\langle N_j\rangle^{-\beta} + \|v_j\|_{X^{\frac{s}{2},\frac{1}{2}+,\delta}}\Big) \tag{6.9}$$

uniformly in $N_j \in \mathbb{Z}_{\ge -1}$, $j = 1, 2, 3$, and $N \ge N_0(\omega, \delta)$ for some $N_0(\omega, \delta) \in \mathbb{N}$. By the definition (2.2) of the local-in-time space, the estimate (6.9) follows once we prove

$$\|\eta_\delta(t) \cdot \mathcal{N}^\omega_{1,N}(\widetilde{v}_1 + z_1, \widetilde{v}_2 + z_2, \widetilde{v}_3 + z_3)\|_{X^{0,-\frac{1}{2}+}} \le C\delta^\theta \prod_{j=1}^{3} \Big(\langle N_j\rangle^{-\beta} + \|\widetilde{v}_j\|_{X^{\frac{s}{2},\frac{1}{2}+}}\Big) \tag{6.10}$$

for any extension $\widetilde{v}_j$ of v_j (restricted to the time interval $[-\delta, \delta]$) onto $\mathbb{R}$, $j = 1, 2, 3$. For simplicity of notations, we denote the extension $\widetilde{v}_j$ by v_j in the following.

By duality, we have

$$\text{LHS of (6.10)} = \sup_{\|a\|_{X^{0,\frac{1}{2}-}} \le 1} \bigg| \int_{\mathbb{T}\times\mathbb{R}} \eta_\delta(t) \cdot \mathcal{N}^\omega_{1,N}(v_1 + z_1, v_2 + z_2, v_3 + z_3)\overline{a(x,t)}dxdt\bigg|, \tag{6.11}$$

where η_δ is as in (2.3). By (4.12) and expanding the product, we write the double integral in (6.11) as[15]

$$\begin{aligned}
\int_{\mathbb{R}} \eta_\delta(t) \sum_n \sum_{\Gamma(n)} e^{it\Psi^\omega_N(\bar{n})} \Big[& \widehat{v}_1(n_1)\overline{\widehat{v}_2(n_2)}\widehat{v}_3(n_3)\overline{\widehat{a}(n)} \\
& + \mathbf{1}_{|n_1|>N_1}(n_1) e^{-itn_1^4} g_{n_1} \overline{\widehat{v}_2(n_2)}\widehat{v}_3(n_3)\overline{\widehat{a}(n)} + \text{ similar terms} \\
& + \bigg(\prod_{j=1}^{2} \mathbf{1}_{|n_j|>N_j}\bigg) e^{-it(n_1^4 - n_2^4)} g_{n_1}\overline{g_{n_2}}\widehat{v}_3(n_3)\overline{\widehat{a}(n)} + \text{ similar terms} \\
& + \bigg(\prod_{j=1}^{3} \mathbf{1}_{|n_j|>N_j}\bigg) e^{-it(n_1^4 - n_2^4 + n_3^4)} g_{n_1}\overline{g_{n_2}}g_{n_3}\overline{\widehat{a}(n)}\Big] dt \\
=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}, &
\end{aligned}$$

where the term I consists of the term with all three factors given by v_j's, II consists of the terms with one factor of z_j and two factors of v_j's, III consists of the terms with two factors of z_j's and one factor of v_j, and IV consists of the term with all three factors given by z_j's.

• **Estimate on I.** Define

$$b_n^{(j)} = e^{itn^4 + it|g_n^N|^2}\langle n\rangle^s \widehat{v}_j(n) \qquad \text{and} \qquad a_n = e^{itn^4 + it|g_n^N|^2}\langle n\rangle^{2s}\widehat{a}(n), \tag{6.12}$$

[15]Here and in the following, we suppress the time dependence.

essentially representing the Fourier transforms of the ungauged interaction representations of v_j and a. Then, we have

$$\begin{aligned}
\mathrm{I} &= \int_{\mathbb{R}} \eta_\delta(t) \sum_n \sum_{\Gamma(n)} e^{it\Psi_N^\omega(\bar{n})} \widehat{v}_1(n_1)\overline{\widehat{v}_2(n_2)}\widehat{v}_3(n_3)\overline{\widehat{a}(n)}\, dt \\
&= \sum_n \sum_{\Gamma(n)} \frac{1}{\langle n_1\rangle^s \langle n_2\rangle^s \langle n_3\rangle^s \langle n\rangle^{2s}} \int_{\mathbb{R}} \big(\eta_\delta(t) e^{-it\Phi(\bar{n})}\big)\, b_{n_1}^{(1)}\overline{b_{n_2}^{(2)}} b_{n_3}^{(3)}\overline{a_n}\, dt.
\end{aligned}$$

By Parseval's identity in the t variable, we have

$$\mathrm{I} = \sum_n \sum_{\Gamma(n)} \frac{1}{\langle n_1\rangle^s \langle n_2\rangle^s \langle n_3\rangle^s \langle n\rangle^{2s}} \int_{\mathbb{R}} \widehat{\eta}_\delta(\tau + \Phi(\bar{n}))\mathcal{F}(b_{n_1}^{(1)}\overline{b_{n_2}^{(2)}} b_{n_3}^{(3)}\overline{a_n})(-\tau) d\tau.$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathrm{I} &\lesssim \bigg(\sum_n \sum_{\Gamma(n)} \frac{1}{\langle n_1\rangle^{2s} \langle n_2\rangle^{2s} \langle n_3\rangle^{2s} \langle n\rangle^{4s}} \bigg\| \frac{\widehat{\eta}_\delta(\tau+\Phi(\bar{n}))}{\langle \tau\rangle^{\frac{1}{2}-}} \bigg\|_{L^2_\tau}^2 \bigg)^{\frac{1}{2}} \\
&\quad \times \bigg(\sum_n \sum_{\Gamma(n)} \Big\| \langle \tau\rangle^{\frac{1}{2}-} \mathcal{F}(b_{n_1}^{(1)}\overline{b_{n_2}^{(2)}} b_{n_3}^{(3)}\overline{a_n})(\tau) \Big\|_{L^2_\tau}^2 \bigg)^{\frac{1}{2}}. \qquad (6.13)
\end{aligned}$$

By Lemma 2.4 with (2.3), we have

$$\bigg\| \frac{\widehat{\eta}_\delta(\tau - \Phi(\bar{n}))}{\langle \tau\rangle^{\frac{1}{2}-\varepsilon}} \bigg\|_{L^2_\tau} \lesssim \bigg(\int \frac{\delta^2}{\langle \tau\rangle^{1-2\varepsilon}\delta\langle \tau - \Phi(\bar{n})\rangle} d\tau \bigg)^{\frac{1}{2}} \lesssim \frac{\delta^{\frac{1}{2}}}{\langle \Phi(\bar{n})\rangle^{\frac{1}{2}-2\varepsilon}} \qquad (6.14)$$

for any small $\varepsilon > 0$. Then, by (6.14) and Lemma 2.1 , we can bound the first factor of (6.13) by

$$\bigg(\sum_n \sum_{\Gamma(n)} \frac{1}{\langle n_1\rangle^{2s} \langle n_2\rangle^{2s} \langle n_3\rangle^{2s} \langle n\rangle^{4s}} \frac{\delta}{\langle \Phi(\bar{n})\rangle^{1-}} \bigg)^{\frac{1}{2}} \lesssim \delta^{\frac{1}{2}}, \qquad (6.15)$$

provided that $s < 0$ is sufficiently close to 0. Next, we consider the second factor of (6.13). By Lemma 2.5, we have

$$\begin{aligned}
\sum_n \sum_{\Gamma(n)} \Big\| \langle \tau\rangle^{\frac{1}{2}-} \mathcal{F}(b_{n_1}^{(1)}\overline{b_{n_2}^{(2)}} b_{n_3}^{(3)}\overline{a_n})(\tau) \Big\|_{L^2_\tau}^2 &= \sum_n \sum_{\Gamma(n)} \Big\| b_{n_1}^{(1)}\overline{b_{n_2}^{(2)}} b_{n_3}^{(3)}\overline{a_n} \Big\|_{H^{\frac{1}{2}-}}^2 \\
&\lesssim \sum_n \sum_{n_1,n_2,n_3} \|b_{n_1}^{(1)}\|_{H^{\frac{1}{2}+}}^2 \|b_{n_2}^{(2)}\|_{H^{\frac{1}{2}+}}^2 \|b_{n_3}^{(3)}\|_{H^{\frac{1}{2}+}}^2 \|a_n\|_{H^{\frac{1}{2}-}}^2 \\
&= \bigg(\sum_n \|a_n\|_{H^{\frac{1}{2}-}}^2 \bigg) \prod_{j=1}^3 \bigg(\sum_{n_j} \|b_{n_j}^{(j)}\|_{H^{\frac{1}{2}+}}^2 \bigg). \qquad (6.16)
\end{aligned}$$

By (6.12), Plancherel's identity, and Lemma 6.2, we have that

$$\begin{aligned}
\sum_n \|b_n^{(j)}\|_{H^{\frac{1}{2}+}}^2 &= \sum_n \big\| \langle n\rangle^s e^{itn^4 + it|g_n^N|^2} \widehat{v}_j(n) \big\|_{H^{\frac{1}{2}+}}^2 \\
&= \sum_n \langle n\rangle^{2s} \big\| \langle \tau + n^4 + |g_n^N|^2\rangle^{\frac{1}{2}+} \widehat{v}_j(n,\tau) \big\|_{L^2_\tau}^2 \\
&= \|v_j\|_{X_+^{s,\frac{1}{2}+}(\omega,N)}^2 \lesssim \delta^{-\varepsilon} \|v_j\|_{X^{\frac{s}{2},\frac{1}{2}+}}^2 \qquad (6.17)
\end{aligned}$$

and

$$\sum_n \|a_n\|^2_{H^{\frac12-}} = \|a\|^2_{X_+^{2s,\frac12+}(\omega,N)} \lesssim \delta^{-\varepsilon}\|a\|^2_{X^{0,\frac12+}}$$

for small $\varepsilon>0$, outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$. Collecting estimates (6.13), (6.15), (6.16), and (6.17), we obtain

$$\mathrm{I}(\omega) \lesssim \delta^{\frac12-}\prod_{j=1}^3 \|v_j\|_{X^{\frac{s}{2},\frac12+}}$$

outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$.

• **Estimate on II.** Without loss of generality, we may assume II has only one term:

$$\mathrm{II} = \int_{\mathbb{R}} \eta_\delta(t)\sum_n\sum_{\Gamma(n)} e^{it\Psi_N^\omega(\bar n)}\Big[\mathbf{1}_{|n_1|>N_1} e^{-itn_1^4} g_{n_1}\overline{\widehat{v}_2(n_2)}\widehat{v}_3(n_3)\overline{\widehat{a}(n)}\Big]dt.$$

With $b_n^{(j)}$ and a_n as in (6.12), Parseval's identity yields

$$\mathrm{II} = \sum_n\sum_{\Gamma(n)} \frac{\mathbf{1}_{|n_1|>N_1} g_{n_1}}{\langle n_2\rangle^s\langle n_3\rangle^s\langle n\rangle^{2s}}\int_{\mathbb{R}} \widehat{\eta}_\delta(\tau+\Phi(\bar n)-|g^N_{n_1}|^2)\mathcal{F}(\overline{b^{(2)}_{n_2}}b^{(3)}_{n_3}\overline{a_n})(-\tau)d\tau.$$

By Cauchy-Schwarz inequality in τ and then in n, n_2, n_3, we have

$$\begin{aligned}
\mathrm{II} &\le \bigg\|\sum_{\substack{n_1\\(n_1,n_2,n_3)\in\Gamma(n)}} \frac{\mathbf{1}_{|n_1|>N_1} g_{n_1}}{\langle n_2\rangle^s\langle n_3\rangle^s\langle n\rangle^{2s}}\frac{\widehat{\eta}_\delta(\tau+\Phi(\bar n)-|g^N_{n_1}|^2)}{\langle\tau\rangle^{\frac12-}}\bigg\|_{\ell^2_{n,n_2,n_3}L^2_\tau}\\
&\quad\times\Big\|\langle\tau\rangle^{\frac12-}\mathcal{F}(\overline{b^{(2)}_{n_2}}b^{(3)}_{n_3}\overline{a_n})(\tau)\Big\|_{\ell^2_{n,n_2,n_3}L^2_\tau}\\
&\lesssim S^{s,\frac12-,\delta}_{1,N}(\omega)\Big\|\big\|\overline{b^{(2)}_{n_2}}b^{(3)}_{n_3}\overline{a_n}\big\|_{H_\tau^{\frac12-}}\Big\|_{\ell^2_{n,n_2,n_3}}.
\end{aligned}$$

where $S^{s,b,\delta}_{1,N}(\omega)$ is defined in (6.6). Proceeding as in (6.16) and (6.17), we arrive at

$$\mathrm{II} \le S^{s,\frac12-,\delta}_{1,N}(\omega)\|v_2\|_{X_+^{s,\frac12+}(\omega,N)}\|v_3\|_{X_+^{s,\frac12+}(\omega,N)}\|a\|_{X_+^{2s,\frac12-}(\omega,N)},$$

Then, by applying Lemmas 6.1 and 6.2, we conclude that there exist small $\theta,\beta>0$ and $s<0$ close to 0 such that

$$\mathrm{II}(\omega) \lesssim \delta^\theta\langle N_1\rangle^{-\beta}\prod_{j=2}^3\|v_j\|_{X^{\frac{s}{2},\frac12+}}$$

outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$.

• **Estimate on III.** Without loss of generality, we assume that III has the following form:

$$\mathrm{III} = \int_{\mathbb{R}}\eta_\delta(t)\sum_n\sum_{\Gamma(n)} e^{it\Psi_N^\omega(\bar n)}e^{-it(n_1^4-n_2^4)}\chi_{1,2}\cdot g_{n_1}\overline{g_{n_2}}\widehat{v}_3(n_3)\overline{\widehat{a}(n)}dt$$

where $\chi_{1,2} := \prod_{j=1}^2 \mathbf{1}_{|n_j|>N_j}$. By Parseval's identity as before, we have

$$\mathrm{III} = \sum_n\sum_{\Gamma(n)}\frac{\chi_{1,2}\cdot g_{n_1}\overline{g_{n_2}}}{\langle n_3\rangle^s\langle n\rangle^{2s}}\int_{\mathbb{R}}\frac{\widehat{\eta}_\delta(\tau+\Phi(\bar n)-|g^N_{n_1}|^2+|g^N_{n_2}|^2)}{\langle\tau\rangle^{\frac12-}}\Big(\langle\tau\rangle^{\frac12-}\mathcal{F}(b^{(3)}_{n_3}\overline{a_n})(-\tau)\Big)d\tau,$$

where $b_n^{(3)}$ and a_n are as in (6.12). By Cauchy-Schwarz inequality and proceeding as before we obtain

$$\begin{aligned}
\mathrm{III} &\leq \Bigg\| \sum_{\substack{n_1,n_2 \\ (n_1,n_2,n_3)\in\Gamma(n)}} \chi_{1,2}\cdot g_{n_1}\overline{g_{n_2}} \frac{\widehat{\eta}_\delta(\tau+\Phi(\bar n)-|g^N_{n_1}|^2+|g^N_{n_2}|^2)}{\langle n_3\rangle^s\langle n\rangle^{2s}\langle\tau\rangle^{\frac12-}} \Bigg\|_{\ell^2_{n,n_3}L^2_\tau} \\
&\quad\times \Big\|\langle\tau\rangle^{\frac12-}\mathcal{F}(b^{(3)}_{n_3}a_n)(\tau)\Big\|_{\ell^2_{n,n_3}L^2_\tau} \\
&\lesssim S^{s,\frac12-,\delta}_{2,N}(\omega)\|v_3\|_{X^{s,\frac12+}_+(\omega,N)}\|a\|_{X^{2s,\frac12-}_+(\omega,N)}
\end{aligned}$$

where $S^{s,b,\delta}_{2,N}(\omega)$ is defined in (6.7). Then, by applying Lemmas 6.1 and 6.2 we conclude that there exist small $\theta,\beta>0$ and $s<0$ close to 0 such that

$$\mathrm{III}(\omega)\lesssim \delta^\theta\bigg(\prod_{j=1}^{2}\langle N_1\rangle^{-\beta}\bigg)\|v_3\|_{X^{\frac{s}{2},\frac12+}}$$

outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$.

• **Estimate on IV.** Lastly, we consider IV. We have

$$\begin{aligned}
\mathrm{IV} &= \int_{\mathbb{R}}\eta_\delta(t)\sum_n\sum_{\Gamma(n)} e^{it\Psi^\omega_N(\bar n)}e^{-it(n_1^4-n_2^4+n_3^4)}\chi_{1,2,3}\cdot g_{n_1}\overline{g_{n_2}}g_{n_3}\overline{\widehat{a}(n)}dt \\
&= \sum_n\sum_{\Gamma(n)}\frac{\chi_{1,2,3}\cdot g_{n_1}\overline{g_{n_2}}g_{n_3}}{\langle n\rangle^{2s}}\int_{\mathbb{R}}\eta_\delta(t)e^{it(\Psi^\omega_{3,N}(\bar n)-\Phi(\bar n))}\overline{a_n}dt,
\end{aligned}$$

where $\chi_{1,2,3}:=\prod_{j=1}^3\mathbf{1}_{|n_j|>N_j}$, Ψ^ω_N is as in (4.9), and $\Psi_{3,N}:=|g^N_{n_1}|^2-|g^N_{n_2}|^2+|g^N_{n_3}|^2$. By applying Parseval's identity and Cauchy-Schwarz inequality as before, we have

$$\begin{aligned}
\mathrm{IV} &= \sum_n\sum_{\Gamma(n)}\frac{\chi_{1,2,3}\cdot g_{n_1}\overline{g_{n_2}}g_{n_3}}{\langle n\rangle^{2s}}\int_{\mathbb{R}}\widehat{\eta}_\delta(\tau+\Phi(\bar n)-\Psi^\omega_{3,N}(\bar n))\overline{\widehat{a_n}(\tau)}d\tau \\
&\leq \Bigg\|\sum_{\Gamma(n)}\chi_{1,2,3}\cdot g_{n_1}\overline{g_{n_2}}g_{n_3}\frac{\widehat{\eta}_\delta(\tau+\Phi(\bar n)-\Psi^\omega_{3,N}(\bar n))}{\langle n\rangle^{2s}\langle\tau\rangle^{\frac12-}}\Bigg\|_{\ell^2_nL^2_\tau}\Big\|\|a_n\|_{H^{\frac12-}}\Big\|_{\ell^2_n} \\
&\leq S^{s,\frac12-,\delta}_{3,N}(\omega)\|a\|_{X^{2s,\frac12-}_+(\omega,N)},
\end{aligned}$$

where $S^{s,b,\delta}_{3,N}(\omega)$ is defined in (6.8). Then, by applying Lemmas 6.1 and 6.2 we conclude that there exist small $\theta,\beta>0$ and $s<0$ close to 0 such that

$$\mathrm{IV}(\omega)\lesssim\delta^\theta\prod_{j=1}^{3}\langle N_1\rangle^{-\beta}$$

outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$.

This completes the proof of Proposition 4.1.

7. Nonlinear estimate II: resonant part

This section is devoted to the proof of Proposition 4.2. Recall from (4.16) and (4.17) that

$$\widetilde{\mathcal{I}}^{\omega}_{2,N}(w_1,w_2,w_3,w_4,w_5)(x,t) = \int_0^t S(t-t')\sum_{n\in\mathbb{Z}} e^{inx}\mathcal{E}^N_n(w_1,w_2,w_3,w_4)(t')\widehat{w}_5(n,t')dt',$$

where

$$\mathcal{E}^N_n(w_1,w_2,w_3,w_4)(t) = -2\operatorname{Re} i\int_0^t \sum_{\Gamma(n)} e^{it'\Psi^{\omega}_N(\bar{n})}\widehat{w}_1(n_1,t')\overline{\widehat{w}_2(n_2,t')}\widehat{w}_3(n_3,t')\overline{\widehat{w}_4(n,t')}dt'.$$

Given w_j, let $v_j = w_j - S(t)\pi^{\perp}_{N_j}(u^{\omega}_0)$. Then, we denote by $\widetilde{v}_j$ an extension of v_j (viewed as a function on the time interval $[-\delta,\delta]$) and set

$$\widetilde{w}_j = S(t)\pi^{\perp}_{N_j}(u^{\omega}_0) + \widetilde{v}_j.$$

Let $s<0<\beta$ be sufficiently close to 0. By the linear estimate (Lemma 2.3) and the definition (2.2) of the local-in-time space, it suffices to construct $\Omega_\delta\subset\Omega$ with $P(\Omega^c_\delta) < e^{-\frac{1}{\delta^c}}$ such that for each $\omega\in\Omega_\delta$, we have

$$\begin{aligned}\bigg\|\chi_\delta(t)\sum_{n\in\mathbb{Z}} e^{inx}\mathcal{E}^N_n(\widetilde{w}_1,\widetilde{w}_2,\widetilde{w}_3,\widetilde{w}_4)(t)\widehat{\widetilde{w}_5}(n,t)\bigg\|_{X^{0,-\frac{1}{2}+}}&\\ \le C\delta^{\theta}\prod_{j=1}^{5}\Big(\langle N_j\rangle^{-\beta} + \|\widetilde{v}_j\|_{X^{\frac{s}{2},\frac{1}{2}+}}\Big)&\end{aligned} \tag{7.1}$$

for any extension $\widetilde{v}_j$ of v_j, $j=1,\dots,5$, uniformly in $N_j\in\mathbb{Z}_{\ge -1}$, $j=1,\dots,5$, and $N\ge N_0(\omega,\delta)$ for some $N_0(\omega,\delta)\in\mathbb{N}$. For simplicity of notations, we denote $\widetilde{v}_j$ (and $\widetilde{w}_j$, respectively) by v_j (and w_j, respectively) in the following. We also suppress the time dependence when it is clear from the context.

By the (continuous) trivial embedding $L^2(\mathbb{T}\times\mathbb{R}) = X^{0,0}\subset X^{0,-\frac{1}{2}+}$ and Hölder's inequality, we have

$$\begin{aligned}\text{LHS of (7.1)} &\lesssim \bigg\|\chi_\delta(t)\bigg(\sum_{n\in\mathbb{Z}}|\mathcal{E}^N_n(w_1,w_2,w_3,w_4)(t)\widehat{w}_5(n,t)|^2\bigg)^{\frac{1}{2}}\bigg\|_{L^2_t}\\ &\lesssim \delta^{\frac{1}{2}}\sup_{t\in[-\delta,\delta]}\bigg(\sum_{n\in\mathbb{Z}}|\mathcal{E}^N_n(w_1,w_2,w_3,w_4)(t)\widehat{w}_5(n,t)|^2\bigg)^{\frac{1}{2}}.\end{aligned}$$

Therefore, in order to prove Proposition 4.2, it suffices to prove

$$\begin{aligned}\sup_{t\in[-\delta,\delta]}\bigg(\sum_{n\in\mathbb{Z}}|\mathcal{E}^N_n(w_1,w_2,w_3,w_4)(t)\widehat{w}_5(n,t)|^2\bigg)^{\frac{1}{2}}&\\ \le C\delta^{-\frac{1}{2}+}\prod_{j=1}^{5}\Big(\langle N_j\rangle^{-\beta} + \|v_j\|_{X^{\frac{s}{2},\frac{1}{2}+}}\Big)&\end{aligned} \tag{7.2}$$

with large probability, where v_j is given by

$$v_j = w_j - S(t)\pi^{\perp}_{N_j}(u^{\omega}_0).$$

Step (i): Elimination of w_5. With $s<0$ close to 0, we have

$$\begin{aligned}&\bigg(\sum_{n\in\mathbb{Z}}|\mathcal{E}_n^N(w_1,w_2,w_3,w_4)\widehat{w}_5(n)|^2\bigg)^{\frac12}\\&\qquad\le\bigg(\sum_{n\in\mathbb{Z}}\langle n\rangle^{-2s}|\mathcal{E}_n^N(w_1,w_2,w_3,w_4)|^2\bigg)^{\frac12}\cdot\sup_n\big|\langle n\rangle^s\mathbf{1}_{|n|>N_5}g_n(\omega)\big|\\&\qquad\quad+\bigg(\sum_{n\in\mathbb{Z}}\langle n\rangle^{-2s}|\mathcal{E}_n^N(w_1,w_2,w_3,w_4)|^2\bigg)^{\frac12}\cdot\sup_n|\langle n\rangle^s\widehat{v}_5(n)|.\end{aligned}\tag{7.3}$$

By applying Lemma 2.7 with $\varepsilon=-\frac{s}{2}>0$, we conclude that

$$\sup_n\big|\langle n\rangle^s\mathbf{1}_{|n|>N_5}g_n(\omega)\big|\le\langle N_5\rangle^{\frac{s}{2}}\delta^{0-},\tag{7.4}$$

outside an exceptional set of probability $<Ce^{-\frac{1}{\delta^c}}$. We also have

$$\sup_{t\in[-\delta,\delta]}\sup_n|\langle n\rangle^s\widehat{v}_5(n,t)|\lesssim\|v_5\|_{X^{s,\frac12+}}.\tag{7.5}$$

Therefore, we conclude from (7.3), (7.4), and (7.5) that, in order to prove (7.2), it suffices to show the following estimate:

$$\sup_{t\in[-\delta,\delta]}\bigg(\sum_{n\in\mathbb{Z}}\langle n\rangle^{-2s}|\mathcal{E}_n^N(w_1,w_2,w_3,w_4)|^2\bigg)^{\frac12}\le C\delta^{-\frac12+}\prod_{j=1}^4\Big(\langle N_j\rangle^{-\beta}+\|v_j\|_{X^{\frac{s}{2},\frac12+}}\Big)\tag{7.6}$$

outside an exceptional set of probability $<Ce^{-\frac{1}{\delta^c}}$, uniformly in $N_j\in\mathbb{Z}_{\ge-1}$, $j=1,\dots,5$, and $N\ge N_0(\omega,\delta)$ for some $N_0(\omega,\delta)\in\mathbb{N}$.

Step (ii) Smoothing effect. In the remaining part of this section, we present the proof of (7.6). By expanding the product of

$$\widehat{w}_j(n_j,t)=\widehat{v}_j(n_j,t)+e^{-itn_j^4}\mathbf{1}_{|n_j|>N_j}g_{n_j},$$

we can bound the left-hand side of (7.6) (without the supremum in time) by

$$\begin{aligned}&\bigg\|\langle n\rangle^{-s}\int_0^t\sum_{\Gamma(n)}e^{it'\Psi_N^\omega(\bar n)}\widehat{w}_1(n_1,t')\overline{\widehat{w}_2(n_2,t')}\widehat{w}_3(n_3,t')\overline{\widehat{w}_4(n,t')}dt'\bigg\|_{\ell_n^2}\\&\lesssim A+B+C+D+E,\end{aligned}\tag{7.7}$$

where A, B, C, D, and E are given by

$$A := \bigg\| \langle n \rangle^{-s} \int_0^t \sum_{\Gamma(n)} e^{it'(\Psi_N^\omega(\bar{n}) - \Phi(\bar{n}))} \chi_{1,2,3.4} \cdot g_{n_1} \overline{g_{n_2}} g_{n_3} \overline{g_n} dt' \bigg\|_{\ell^2_n},$$

$$B := \bigg\| \langle n \rangle^{-2s} \int_0^t \sum_{\Gamma(n)} e^{it'(\Psi_{3,N}^\omega(\bar{n}) - \Phi(\bar{n}))} \chi_{1,2,3} \cdot g_{n_1} \overline{g_{n_2}} g_{n_3} \overline{b_n^{(4)}} dt' \bigg\|_{\ell^2_n} + \text{similar terms},$$

$$C := \bigg\| \int_0^t \sum_{\Gamma(n)} \frac{e^{it'(\Psi_{2,N}^\omega(\bar{n}) - \Phi(\bar{n}))}}{\langle n_3 \rangle^s \langle n \rangle^{2s}} \chi_{1,2} \cdot g_{n_1} \overline{g_{n_2}} b_{n_3}^{(3)} \overline{b_n^{(4)}} dt' \bigg\|_{\ell^2_n} + \text{similar terms},$$

$$D := \bigg\| \int_0^t \sum_{\Gamma(n)} \frac{e^{it'(|g_{n_1}^N|^2 - \Phi(\bar{n}))}}{\langle n_2 \rangle^s \langle n_3 \rangle^s \langle n \rangle^{2s}} \chi_1 \cdot g_{n_1} \overline{b_{n_2}^{(2)}} b_{n_3}^{(3)} \overline{b_n^{(4)}} dt' \bigg\|_{\ell^2_n} + \text{similar terms},$$

$$E := \bigg\| \int_0^t \sum_{\Gamma(n)} \frac{e^{-it'\Phi(\bar{n})}}{\langle n_1 \rangle^s \langle n_2 \rangle^s \langle n_3 \rangle^s \langle n \rangle^{2s}} b_{n_1}^{(1)} \overline{b_{n_2}^{(2)}} b_{n_3}^{(3)} \overline{b_n^{(4)}} dt' \bigg\|_{\ell^2_n}.$$

Here, $b_n^{(j)}$ is as in (6.12),

$$\chi_{1,\dots,k} = \prod_{j=1}^k \mathbf{1}_{|n_j| > N_j}, \quad k = 1, \dots, 4,$$

and

$$\Psi_{k,N}^\omega(\bar{n}) = \sum_{j=1}^k (-1)^{j+1} |g_{n_j}^N|^2, \quad k = 2, 3.$$

In view of the restriction of the time variable onto $[-\delta, \delta]$, we may freely insert the cutoff functions $\chi_\delta(t)$ and $\eta_\delta(t)$ in evaluating the terms A, B, C, D, and E. In the following, we prove (7.6) by estimating each term on the right-hand side of (7.7).

(ii.1) Estimate on A. Fix $\kappa, \varepsilon > 0$ small. By applying Lemma 2.7, we have

$$|g_n(\omega)| \lesssim \delta^{-\frac{\kappa}{2}} \langle n \rangle^\varepsilon \tag{7.8}$$

outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$. Then, for such ω, we split $A(\omega)$ into two parts:

$$A(\omega) = A_1(\omega) + A_2(\omega),$$

where $A_1(\omega)$ denotes the contribution from the case $n_{\max} \lesssim \delta^{-\kappa}$. Namely, we have

$$A_1(\omega) := \bigg\| \langle n \rangle^{-s} \chi_\delta(t) \int_0^t \sum_{\Gamma(n)} \mathbf{1}_{n_{\max} \lesssim \delta^{-\kappa}} e^{it'(\Psi_N^\omega(\bar{n}) - \Phi(\bar{n}))} \chi_{1,2,3.4} \cdot g_{n_1} \overline{g_{n_2}} g_{n_3} \overline{g_n} dt' \bigg\|_{\ell^2_n}.$$

Note that if $\max(N_1, N_2, N_3, N_4) \gg \delta^{-\kappa}$, then we have $A_1(\omega) = 0$. Otherwise, using (7.8), we have

$$A_1(\omega) \lesssim \delta^{1 + s\kappa - C\kappa} \prod_{j=1}^4 \langle N_j \rangle^{-1}$$

for some $C > 0$. This yields (7.6).

Next, we consider $A_2(\omega)$. Since $n_{\max} \gg \delta^{-\kappa}$, we have $|g_n(\omega)| \lesssim \delta^{-\frac{\kappa}{2}} \langle n \rangle^\varepsilon \ll n_{\max}^{\frac{1}{2}+\varepsilon}$. Then, it follows from Lemma 2.1 and (6.1), we have

$$|\Psi_N^\omega(\bar{n}) - \Phi(\bar{n})| \sim \langle \Phi(\bar{n}) \rangle \tag{7.9}$$

for $(n_1, n_2, n_3) \in \Gamma(n)$. Thus, from (7.8), (7.9), and Lemma 2.1, we obtain

$$\begin{aligned}
A_2(\omega) &= \bigg\| \langle n \rangle^{-s} \sum_{\Gamma(n)} \frac{e^{it(\Psi_N^\omega(\bar{n})-\Phi(\bar{n}))} - 1}{\Psi_N^\omega(\bar{n}) - \Phi(\bar{n})} \chi_{1,2,3.4} \cdot g_{n_1} \overline{g_{n_2}} g_{n_3} \overline{g_n} \bigg\|_{\ell_n^2} \\
&\lesssim \delta^{-2\kappa} \bigg(\prod_{j=1}^4 \langle N_j \rangle^{-\beta} \bigg) \bigg\| \sum_{\Gamma(n)} \frac{n_{\max}^{4\beta+4\varepsilon-s}}{\langle \Phi(\bar{n}) \rangle} \chi_{1,2,3.4} \bigg\|_{\ell_n^2} \\
&\lesssim \delta^{-2\kappa} \bigg(\prod_{j=1}^4 \langle N_j \rangle^{-\beta} \bigg) \bigg\| \sum_{\Gamma(n)} \frac{1}{n_{\max}^{2-4\beta-4\varepsilon+s}(n-n_1)(n-n_3)} \bigg\|_{\ell_n^2} \\
&\lesssim \delta^{-2\kappa} \prod_{j=1}^4 \langle N_j \rangle^{-\beta},
\end{aligned}$$

provided that $\varepsilon, \beta, -s > 0$ are sufficiently small. This yields (7.6).

(ii.2) Estimate on B. Without loss of generality, we may assume that B consists only of one term:

$$B = \bigg\| \langle n \rangle^{-2s} \int_0^t \sum_{\Gamma(n)} e^{it'(\Psi_{3,N}^\omega(\bar{n})-\Phi(\bar{n}))} \chi_{1,2,3} \cdot g_{n_1} \overline{g_{n_2}} g_{n_3} \overline{b_n^{(4)}} dt' \bigg\|_{\ell_n^2}.$$

To exploit the oscillatory nature of the time integral, we rewrite the above integral as

$$\int_{\mathbb{R}} \eta_\delta(t') \sum_{\Gamma(n)} e^{it'(\Psi_{3,N}^\omega(\bar{n})-\Phi(\bar{n}))} \chi_{1,2,3} \cdot g_{n_1} \overline{g_{n_2}} g_{n_3} \Big(\mathbf{1}_{[0,t]}(t') \overline{b_n^{(4)}(t')} \Big) dt',$$

where η_δ is as in (2.3). Then, by Parseval's identity, the above expression is

$$\begin{aligned}
&\sum_{\Gamma(n)} \chi_{1,2,3} \cdot g_{n_1} \overline{g_{n_2}} g_{n_3} \int_{\mathbb{R}} \widehat{\eta}_\delta(\tau + \Phi(\bar{n}) - \Psi_{3,N}^\omega(\bar{n})) \mathcal{F}_t(\mathbf{1}_{[0,t]} \overline{b_n^{(4)}})(-\tau) d\tau \\
&\quad = \sum_{\Gamma(n)} \chi_{1,2,3} \cdot g_{n_1} \overline{g_{n_2}} g_{n_3} \int_{\mathbb{R}} \frac{\widehat{\eta}_\delta(\tau + \Phi(\bar{n}) - \Psi_{3,N}^\omega(\bar{n}))}{\langle \tau \rangle^{\frac{1}{2}-}} \\
&\qquad\qquad \times \Big(\langle \tau \rangle^{\frac{1}{2}-} \mathcal{F}_t(\mathbf{1}_{[0,t]} \overline{b_n^{(4)}})(-\tau) \Big) d\tau.
\end{aligned}$$

Therefore, by Cauchy-Schwarz inequality in the τ variable and Lemma 2.6, we have

$$\begin{aligned}
B &\le \bigg\| \sum_{\Gamma(n)} \chi_{1,2,3} \cdot g_{n_1} \overline{g_{n_2}} g_{n_3} \frac{\widehat{\eta}_\delta(\tau + \Phi(\bar{n}) - \Psi_{3,N}^\omega(\bar{n}))}{\langle n \rangle^{2s} \langle \tau \rangle^{\frac{1}{2}-}} \bigg\|_{\ell_n^2 L_\tau^2} \Big\| \| \mathbf{1}_{[0,t]}(t') b_n^{(4)}(t') \|_{H_{t'}^{\frac{1}{2}-}} \Big\|_{\ell_n^\infty} \\
&\lesssim S_{3,N}^{s,\frac{1}{2}-,\delta}(\omega) \Big\| \| b_n^{(4)} \|_{H_t^{\frac{1}{2}-}} \Big\|_{\ell_n^\infty},
\end{aligned}$$

where $S_{3,N}^{s,b,\delta}(\omega)$ is defined in (6.8). Then, proceeding as in (6.17), we obtain

$$B(\omega) \lesssim S_{3,N}^{s,\frac{1}{2}-,\delta}(\omega) \|v_4\|_{X_+^{s,\frac{1}{2}-}(\omega,N)}. \tag{7.10}$$

Finally, by applying Lemmas 6.1 and 6.2 to (7.10), we obtain the desired estimate (7.6) for the term B outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$.

(ii.3) Estimate on C. Without loss of generality, we assume that C consists only of one term:

$$C = \bigg\| \int_{\mathbb{R}} \sum_{\Gamma(n)} \eta_\delta(t') \frac{e^{it'(\Psi^\omega_{2,N}(\bar{n}) - \Phi(\bar{n}))}}{\langle n_3\rangle^s \langle n\rangle^{2s}} \chi_{1,2} \cdot g_{n_1}\overline{g_{n_2}} \Big(\mathbf{1}_{[0,t]}(t') b^{(3)}_{n_3}(t') \overline{b^{(4)}_n(t')} \Big) dt' \bigg\|_{\ell^2_n}.$$

By Parseval's identity, we have

$$\begin{aligned} C = \bigg\| \sum_{\Gamma(n)} \chi_{1,2} \cdot g_{n_1}\overline{g_{n_2}} \int_{\mathbb{R}} & \frac{\widehat{\eta}_\delta(\tau + \Phi(\bar{n}) - \Psi^\omega_{2,N}(\bar{n}))}{\langle n_3\rangle^s \langle n\rangle^{2s} \langle \tau\rangle^{\frac{1}{2}-}} \\ & \times \Big(\langle \tau\rangle^{\frac{1}{2}-} \mathcal{F}_t(\mathbf{1}_{[0,t]} b^{(3)}_{n_3} \overline{b^{(4)}_n})(-\tau) \Big) d\tau \bigg\|_{\ell^2_n}. \end{aligned}$$

By Cauchy-Schwarz inequality in τ and n_3 followed by Hölder's inequality in n, we have

$$\begin{aligned} C \le & \bigg\| \sum_{\substack{n_1, n_2 \\ (n_1,n_2,n_3)\in\Gamma(n)}} \chi_{1,2} \cdot g_{n_1}\overline{g_{n_2}} \frac{\widehat{\eta}_\delta(\tau + \Phi(\bar{n}) - \Psi^\omega_{2,N}(\bar{n}))}{\langle n_3\rangle^s \langle n\rangle^{2s} \langle \tau\rangle^{\frac{1}{2}-}} \bigg\|_{\ell^2_{n,n_3} L^2_\tau} \\ & \times \sup_{n\in\mathbb{Z}} \Big\| \| \mathbf{1}_{[0,t]}(t') b^{(3)}_{n_3}(t') \overline{b^{(4)}_n(t')} \|_{H^{\frac{1}{2}-}_{t'}} \Big\|_{\ell^2_{n_3}}. \end{aligned} \tag{7.11}$$

As for the second factor of (7.11), by applying Lemma 2.6 and then Lemma 2.5 and proceeding as in (6.17), we have

$$\begin{aligned} \sup_{n\in\mathbb{Z}} \Big\| \| \mathbf{1}_{[0,t]}(t') b^{(3)}_{n_3}(t') \overline{b^{(4)}_n}(t') \|_{H^{\frac{1}{2}-}_{t'}} \Big\|_{\ell^2_{n_3}} & \lesssim \Big\| \| b^{(3)}_{n_3} \|_{H^{\frac{1}{2}+}} \| b^{(4)}_n \|_{H^{\frac{1}{2}+}} \Big\|_{\ell^2_{n,n_3}} \\ & \lesssim \| v_3 \|_{X^{s,\frac{1}{2}+}_+(\omega,N)} \| v_4 \|_{X^{s,\frac{1}{2}+}_+(\omega,N)}. \end{aligned} \tag{7.12}$$

Therefore, from (7.11) and (7.12), we obtain

$$C(\omega) \lesssim S^{s,\frac{1}{2}-,\delta}_{2,N}(\omega) \| v_3 \|_{X^{s,\frac{1}{2}+}_+(\omega,N)} \| v_4 \|_{X^{s,\frac{1}{2}+}_+(\omega,N)}, \tag{7.13}$$

where $S^{s,b,\delta}_{2,N}(\omega)$ is defined in (6.7). Finally, by applying Lemmas 6.1 and 6.2 to (7.13), we obtain the desired estimate (7.6) for the term C outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$.

(ii.4) Estimate on D. Without loss of generality, we assume that D has only one term:

$$D = \bigg\| \int_0^t \sum_{\Gamma(n)} \eta_\delta(t') \frac{e^{it'(|g^N_{n_1}|^2 - \Phi(\bar{n}))}}{\langle n_2\rangle^s \langle n_3\rangle^s \langle n\rangle^{2s}} \chi_1 \cdot g_{n_1} \Big(\mathbf{1}_{[0,t]}(t') \overline{b^{(2)}_{n_2}(t')} b^{(3)}_{n_3}(t') \overline{b^{(4)}_n(t')} \Big) dt' \bigg\|_{\ell^2_n}.$$

Proceeding as before with Parseval's identity and Hölder's inequality, we have

$$\begin{aligned} D \le & \bigg\| \sum_{\substack{n_1\in\mathbb{Z} \\ (n_1,n_2,n_3)\in\Gamma(n)}} \chi_1 \cdot g_{n_1} \frac{\widehat{\eta}_\delta(\tau + \Phi(\bar{n}) - |g^N_{n_1}|^2)}{\langle n_2\rangle^s \langle n_3\rangle^s \langle n\rangle^{2s} \langle \tau\rangle^{\frac{1}{2}-}} \bigg\|_{\ell^2_{n,n_2,n_3} L^2_\tau} \\ & \times \sup_{n\in\mathbb{Z}} \Big\| \| \mathbf{1}_{[0,t]}(t') \overline{b^{(2)}_{n_2}(t')} b^{(3)}_{n_3}(t') \overline{b^{(4)}_n(t')} \|_{H^{\frac{1}{2}-}_{t'}} \Big\|_{\ell^2_{n_2,n_3}}. \end{aligned}$$

Then, by estimating the the second factor as in (7.12) with Lemmas 2.5 and 2.6, we obtain

$$D(\omega) \lesssim S_{1,N}^{s,\frac{1}{2}-,\delta}(\omega) \prod_{j=2}^{4} \|v_j\|_{X_+^{s,\frac{1}{2}+}(\omega,N)}, \tag{7.14}$$

where $S_{1,N}^{s,b,\delta}(\omega)$ is defined in (6.6). Finally, by applying Lemmas 6.1 and 6.2 to (7.14), we obtain the desired estimate (7.6) for the term D outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$.

(ii.5) Estimate on E. We have

$$E = \bigg\| \int_0^t \sum_{\Gamma(n)} \eta_\delta(t') \frac{e^{-it'\Phi(\bar{n})}}{\langle n_1\rangle^s \langle n_2\rangle^s \langle n_3\rangle^s \langle n\rangle^{2s}} \Big(\mathbf{1}_{[0,1]}(t') b_{n_1}^{(1)}(t') \overline{b_{n_2}^{(2)}(t')} b_{n_3}^{(3)}(t') \overline{b_n^{(4)}(t')} \Big) dt' \bigg\|_{\ell_n^2}.$$

Proceeding as before with Parseval's identity and Hölder's inequality, we have

$$\begin{aligned} E \le \sup_{n\in\mathbb{Z}} & \bigg\| \frac{\widehat{\eta}_\delta(\tau + \Phi(\bar{n}))}{\langle n_1\rangle^s \langle n_2\rangle^s \langle n_3\rangle^s \langle n\rangle^{2s} \langle\tau\rangle^{\frac{1}{2}-}} \bigg\|_{\ell^2_{\Gamma(n)} L^2_\tau} \\ & \times \Big\| \big\| \mathbf{1}_{[0,t]}(t') b_{n_1}^{(1)}(t') \overline{b_{n_2}^{(2)}(t')} b_{n_3}^{(3)}(t') \overline{b_n^{(4)}(t')} \big\|_{H_{t'}^{\frac{1}{2}-}} \Big\|_{\ell^2_{n,\Gamma(n)}}, \end{aligned} \tag{7.15}$$

where the $\ell^2_{\Gamma(n)}$-norm is defined by

$$\|f_{n_1,n_2,n_3}\|_{\ell^2_{\Gamma(n)}} = \bigg(\sum_{(n_1,n_2,n_3)\in\Gamma(n)} |f_{n_1,n_2,n_3}|^2 \bigg)^{\frac{1}{2}}.$$

By Lemma 2.4 followed by Lemma 2.1, we can bound the first factor on the right-hand side of (7.15) by

$$\begin{aligned} \sup_{n\in\mathbb{Z}} & \bigg\| \frac{\delta\widehat{\eta}(\delta(\tau - \Phi(\bar{n})))}{\langle n_1\rangle^s \langle n_2\rangle^s \langle n_3\rangle^s \langle n\rangle^{2s} \langle\tau\rangle^{\frac{1}{2}-}} \bigg\|_{\ell^2_{n_1,n_2,n_3} L^2_\tau} \lesssim \sup_{n\in\mathbb{Z}} \bigg\| \frac{\delta^{\frac{1}{2}}}{n_{\max}^{5s} \langle\tau\rangle^{\frac{1}{2}-} \langle \tau - \Phi(\bar{n})\rangle^{\frac{1}{2}}} \bigg\|_{\ell^2_{\Gamma(n)} L^2_\tau} \\ & \sim \delta^{\frac{1}{2}} \bigg(\sum_{(n_1,n_2,n_3)\in\Gamma(n)} \frac{1}{n_{\max}^{10s} \langle\Phi(\bar{n})\rangle^{1-}} \bigg)^{\frac{1}{2}} \lesssim 1, \end{aligned}$$

provided that $s < 0$ is sufficiently close to 0. The second factor on the right-hand side of (7.15) can be estimated as in (7.12) with Lemmas 2.5 and 2.6. Therefore, we obtain

$$E(\omega) \lesssim \prod_{j=1}^{4} \|v_j\|^2_{X_+^{s,\frac{1}{2}+}(\omega,N)}. \tag{7.16}$$

Finally, by applying Lemma 6.2 to (7.16), we obtain the desired estimate (7.6) for the term E outside an exceptional set of probability $< Ce^{-\frac{1}{\delta^c}}$.

This completes the proof of Proposition 4.2.

Appendix A. Further probabilistic estimates

In this appendix, we state and prove crucial probabilistic estimates. These probabilistic estimates play an important role in establishing Propositions 4.1 and 4.2. In Subsection A.3, we present the proof of Lemma 2.11.

In the following, $\{g_n\}_{n\in\mathbb{Z}}$ denotes a sequence of independent standard complex-valued Gaussian random variables. In particular, we have

$$\mathbb{E}\big[g_n^k\,\overline{g_m^\ell}\big] = \delta_{k\ell}\delta_{nm}\cdot k! \tag{A.1}$$

for any $k,\ell\in\mathbb{Z}_{\geq 0}$ and $n,m\in\mathbb{Z}$. The identity (A.1) easily follows from a computation with the moment generating function for the chi-square distribution of degree 2 (i.e. $|g_n|^2 = (\operatorname{Re} g_n)^2 + (\operatorname{Im} g_n)^2$).

A.1. **Random $X^{s,b}$-space.** Given $N\in\mathbb{N}\cup\{\infty\}$, set $g_n^N = \mathbf{1}_{|n|\leq N}\cdot g_n$ as in (4.8) with the understanding that $\mathbf{1}_{|n|\leq N}\equiv 1$ when $N=\infty$. Then, we define random versions $X_+^{s,b}(\omega,N)$ and $X_-^{s,b}(\omega,N)$ of the $X^{s,b}$-space by the norm:

$$\|u\|_{X_\pm^{s,b}(\omega,N)} = \big\|\langle n\rangle^s\langle\tau+n^4\pm|g_n^N(\omega)|^2\rangle^b\widehat{u}(n,\tau)\big\|_{\ell_n^2L_\tau^2}. \tag{A.2}$$

When $N=\infty$, we simply set $X_{\pm,\omega}^{s,b} = X_\pm^{s,b}(\omega,\infty)$. The following lemma shows that the random $X^{s,b}$-norm is controlled by the standard $X^{s,b}$-norm in (2.1) with large probability.

Lemma A.1. *Let $\eta\in\mathcal{S}(\mathbb{R})$ be a Schwartz function in time and $u\in X^{s,b}$ with $s\in\mathbb{R}$ and $b>0$. Then, there exists $C>0$ such that*

$$\Big\|\sup_{N\in\mathbb{N}\cup\{\infty\}}\|\eta u\|_{X_\pm^{s,b}(\omega,N)}\Big\|_{L^p(\Omega)} \leq Cp^{b+2}\|u\|_{X^{s,b}} \tag{A.3}$$

for all $p\geq 2$, where the constant is independent of u. As a consequence, there exist $c,C>0$ such that

$$P\bigg(\sup_{N\in\mathbb{N}\cup\{\infty\}}\|\eta u\|_{X_\pm^{s,b}(\omega,N)} > K\|u\|_{X^{s,b}}\bigg) \leq Ce^{-K^{\frac{1}{b+2}}\|u\|_{X^{s,b}}^{-\frac{1}{b+2}}} \tag{A.4}$$

for any $K>0$.

We present the proof of Lemma A.1 at the end of this subsection. While the tail estimate (A.4) holds for each *fixed* $u\in X^{s,b}$, Lemma A.1 does not provide a uniform control in $u\in X^{s,b}$ and hence is not useful in the proof of the main nonlinear estimates (Propositions 4.1 and 4.2). By slightly losing spatial regularity, however, we can control the random $X^{s,b}$-norm by the standard $X^{\sigma,b}$-norm (with $\sigma>s$) *uniformly* in $u\in X^{\sigma,b}$. See Lemma 6.2 above.

Lemma A.2. *Let $\sigma>s$ and $b>0$. Then, for each $K>0$, there exists a set $\Omega_K\subset\Omega$ with $P(\Omega_K^c)<Ce^{-cK^{\frac{1}{b}}}$ such that*

$$\sup_{N\in\mathbb{N}\cup\{\infty\}}\|u\|_{X_\pm^{s,b}(\omega,N)} \lesssim (1+K)\|u\|_{X^{\sigma,b}} \tag{A.5}$$

uniformly in $u\in X^{\sigma,b}$.

Proof. Fix $\varepsilon>0$ sufficiently small such that

$$\sigma\geq s+2b\varepsilon.$$

By Lemma 2.7, there exists Ω_K with $P(\Omega_K^c)<Ce^{-cK^{\frac{1}{b}}}$ such that

$$\begin{aligned}\langle\tau+n^4\pm|g_n^N(\omega)|^2\rangle^b &\lesssim \langle\tau+n^4\rangle^b + |g_n^N(\omega)|^{2b}\\ &\lesssim \langle\tau+n^4\rangle^b + K\langle n\rangle^{2b\varepsilon}.\end{aligned}$$

This implies that

$$\sup_{N\in\mathbb{N}\cup\{\infty\}} \|u\|_{X^{s,b}_{\pm}(\omega,N)} \lesssim \|u\|_{X^{s,b}} + K\|u\|_{X^{\sigma,0}}$$

for each $\omega \in \Omega_K$, uniformly in $u \in X^{\sigma,b}$. Then, the desired estimate (A.5) follows from the monotonicity of the $X^{s,b}$-norm in s and b. □

We now present the proof of Lemma A.1.

Proof of Lemma A.1. Trivially, we have

$$\sup_{N\in\mathbb{N}\cup\{\infty\}} \|\eta u\|_{X^{s,b}_{\pm}(\omega,N)} \le \|\eta u\|_{X^{s,b}} + \|\eta u\|_{X^{s,b}_{\pm}(\omega,\infty)}.$$

Since the multiplication by a smooth cutoff function η is bounded in $X^{s,b}$, the estimate (A.3) follows once we prove

$$\Big\| \|\eta u\|_{X^{s,b}_{\pm,\omega}} \Big\|_{L^p(\Omega)} \le Cp^{b+2}\|u\|_{X^{s,b}}. \tag{A.6}$$

The tail estimate (A.4) follows from applying Lemma 2.9 to (A.3).

Let $v(t) = S(-t)u(t)$ denote the interaction representation of u and set $a_n(\tau) = \widehat{v}(n,\tau)$. Then, we have

$$\mathcal{F}(\eta u)(n,\tau) = \int_{\mathbb{R}} \widehat{\eta}(\tau_1 + n^4)a_n(\tau-\tau_1)d\tau_1. \tag{A.7}$$

From the definition (A.2), (A.7), and the triangle inequality $\langle\tau\rangle^b \lesssim \langle\tau_1\rangle^b + \langle\tau-\tau_1\rangle^b$ for $b \ge 0$, we have

$$\begin{aligned}
\|\eta u\|^2_{X^{s,b}_{\pm,\omega}} &= \sum_n \int_{\mathbb{R}} \langle n\rangle^{2s}\langle\tau\rangle^{2b}|\mathcal{F}(\eta u)(n,\tau - n^4 \mp |g_n|^2)|^2 d\tau \\
&= \sum_n \int_{\mathbb{R}} \langle n\rangle^{2s}\langle\tau\rangle^{2b}\bigg|\int_{\mathbb{R}} \widehat{\eta}(\tau_1 \mp |g_n|^2)a_n(\tau-\tau_1)d\tau_1\bigg|^2 d\tau \\
&\lesssim \sum_n \int_{\mathbb{R}} \langle n\rangle^{2s}\bigg|\int_{\mathbb{R}} \langle\tau_1\rangle^b\widehat{\eta}(\tau_1 \mp |g_n|^2)a_n(\tau-\tau_1)d\tau_1\bigg|^2 d\tau \\
&\quad + \sum_n \int_{\mathbb{R}} \langle n\rangle^{2s}\bigg|\int_{\mathbb{R}} \widehat{\eta}(\tau_1 \mp |g_n|^2)\langle\tau-\tau_1\rangle^b a_n(\tau-\tau_1)d\tau_1\bigg|^2 d\tau \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned} \tag{A.8}$$

Before proceeding further, we claim the following inequality:

$$E_b(\tau) := \Big(\mathbb{E}\Big[\langle\tau\rangle^{bp}|\widehat{\eta}(\tau \mp |g_n|^2)|^p\Big]\Big)^{\frac{1}{p}} \le C(b)\frac{p^{b+2}}{\langle\tau\rangle^2}. \tag{A.9}$$

We first use this estimate to bound I and II in (A.8). We present the proof of (A.9) at the end of this proof.

By Minkowski's integral inequality, (A.9), and Young's inequality, we have

$$\begin{aligned}
\mathbb{E}\Big[\mathrm{I}^{\frac{p}{2}}\Big] &\le \bigg(\sum_n \int_{\mathbb{R}} \langle n\rangle^{2s}\bigg|\int_{\mathbb{R}} E_b(\tau_1)|a_n(\tau-\tau_1)|d\tau_1\bigg|^2 d\tau\bigg)^{\frac{p}{2}} \\
&\lesssim p^{(b+2)p}\bigg(\sum_n \langle n\rangle^{2s}\int_{\mathbb{R}} |a_n(\tau)|^2 d\tau\bigg)^{\frac{p}{2}} = p^{(b+2)p}\|u\|^p_{X^{s,0}}.
\end{aligned} \tag{A.10}$$

Similarly, we have

$$\mathbb{E}\Big[\Pi^{\frac{p}{2}}\Big] \le \bigg(\sum_n \int_{\mathbb{R}} \langle n\rangle^{2s}\bigg|\int_{\mathbb{R}} E_0(\tau_1)\langle \tau-\tau_1\rangle^b |a_n(\tau-\tau_1)|d\tau_1\bigg|^2 d\tau\bigg)^{\frac{p}{2}}$$
$$\lesssim p^{2p}\bigg(\sum_n \langle n\rangle^{2s}\int_{\mathbb{R}}\langle\tau\rangle^{2b}|a_n(\tau)|^2 d\tau\bigg)^{\frac{p}{2}} \lesssim p^{2p}\|u\|_{X^{s,b}}. \tag{A.11}$$

Hence, (A.6) follows from (A.8), (A.10), and (A.11).

It remains to prove (A.9). By the triangle inequality:

$$\langle\tau\rangle \lesssim \langle \tau \mp |g_n|^2\rangle + |g_n|^2$$

and using the rapid decay of $\widehat{\eta}\in\mathcal{S}(\mathbb{R})$, we have

$$\big\|\langle\tau\rangle^b\widehat{\eta}(\tau\mp|g_n|^2)|\big\|_{L^p(\Omega)} \le \langle\tau\rangle^{-2}\big\|\langle\tau\rangle^{b+2}\widehat{\eta}(\tau\mp|g_n|^2)\big\|_{L^p(\Omega)}$$
$$\lesssim \langle\tau\rangle^{-2}\big(1+\|g_n\|^{2(b+2)}_{L^{2(b+2)p}(\Omega)}\big) \lesssim \frac{p^{b+2}}{\langle\tau\rangle^2},$$

yielding (A.9). This completes the proof of Lemma A.1. □

A.2. **Key tail estimates.** In the following, we present the proof of the key tail estimates (Lemma 6.1) in establishing crucial nonlinear estimates (Propositions 4.1 and 4.2). Given $s,b\in\mathbb{R}$, $\delta>0$, and $N\in\mathbb{N}$, we recall the definitions of $S^{s,b,\delta}_{j,N}$, $j=1,2,3$, from (6.2), (6.3), and (6.4) (expressed in slightly different forms via Taylor expansions):

$$S^{s,b,\delta}_{1,N}(f) = \Bigg\| \sum_{\substack{n_1\in\mathbb{Z}\\(n_1,n_2,n_3)\in\Gamma(n)}} \widehat{f}(n_1)\frac{\widehat{\eta}_\delta(\tau+\Phi(\bar n)-|g^N_{n_1}|^2)}{\langle n_2\rangle^s\langle n_3\rangle^s\langle n\rangle^{2s}\langle\tau\rangle^b}\Bigg\|_{\ell^2_{n,n_2,n_3}L^2_\tau}, \tag{A.12}$$

$$S^{s,b,\delta}_{2,N}(f_1,f_2) = \Bigg\| \sum_{k_1,k_2=0}^\infty \sum_{\substack{n_1,n_2\in\mathbb{Z}\\(n_1,n_2,n_3)\in\Gamma(n)}} \widehat{f}_1(n_1)\overline{\widehat{f}_2(n_2)} \times\prod_{j=1}^2\frac{|g^N_{n_j}|^{2k_j}}{k_j!}\cdot\frac{\partial^{k_1+k_2}\widehat{\eta}_\delta(\tau+\Phi(\bar n))}{\langle n_3\rangle^s\langle n\rangle^{2s}\langle\tau\rangle^b}\Bigg\|_{\ell^2_{n,n_3}L^2_\tau},$$

$$S^{s,b,\delta}_{3,N}(f_1,f_2,f_3) = \Bigg\| \sum_{k_1,k_2,k_3=0}^\infty \sum_{\Gamma(n)} \widehat{f}_1(n_1)\overline{\widehat{f}_2(n_2)}\widehat{f}_3(n_3) \times\prod_{j=1}^3\frac{|g^N_{n_j}|^{2k_j}}{k_j!}\cdot\frac{\partial^{k_1+k_2+k_3}\widehat{\eta}_\delta(\tau+\Phi(\bar n))}{\langle n\rangle^{2s}\langle\tau\rangle^b}\Bigg\|_{\ell^2_nL^2_\tau}.$$

Here, $\eta\in C^\infty_c(\mathbb{R})$ denotes a smooth non-negative cutoff function supported on $[-2,2]$ with $\eta\equiv1$ on $[-1,1]$, and the notations $\Gamma(n)$, $\Phi(\bar n)$, and $\Psi^\omega_N(\bar n)$ are as in (1.20), (2.6), and (4.9), respectively. We also recall that there is only one term in the summation over n_1 in (A.12).

For simplicity of notations, we set

$$S^{s,b,\delta}_{1,N}(\omega) := S^{s,b,\delta}_{1,N}(\pi^\perp_{N_1}(u^\omega_0)),$$
$$S^{s,b,\delta}_{2,N}(\omega) := S^{s,b,\delta}_{2,N}(\pi^\perp_{N_1}(u^\omega_0),\pi^\perp_{N_2}(u^\omega_0)),$$
$$S^{s,b,\delta}_{3,N}(\omega) := S^{s,b,\delta}_{3,N}(\pi^\perp_{N_1}(u^\omega_0),\pi^\perp_{N_2}(u^\omega_0),\pi^\perp_{N_3}(u^\omega_0)),$$

where u_0^ω is the white noise in (6.5) and $\pi_{N_j}^\perp$ denotes the frequency projection onto the frequencies $\{|n| > N_j\}$ as in (2.5) with the convention that $\pi_{-1}^\perp = \mathrm{Id}$. With the notations defined above, we have the following tail estimates for these random functionals (Lemma 6.1).

Lemma A.3. *Let $s < 0$, $b < \frac{1}{2}$, and $\beta > 0$ such that s and β are sufficiently close to 0 and b is sufficiently close to $\frac{1}{2}$. Then, there exist $c, \kappa > 0$ and small $\delta_0 > 0$ such that the following statements holds.*

(i) *We have*

$$P\bigg(\bigg\{\omega \in \Omega : \sup_{N\in\mathbb{N}} \sup_{N_1\in\mathbb{Z}_{\geq -1}} \langle N_1\rangle^\beta |S_{1,N}^{s,b,\delta}(\omega)| > \delta^\kappa\bigg\}\bigg) < e^{-\frac{1}{\delta^c}} \tag{A.13}$$

for any $0 < \delta < \delta_0$.

(ii) *Let $k = 2, 3$. Given $0 < \delta < \delta_0$, define the sets $\mathcal{A}_k$ by*

$$\mathcal{A}_k := \bigg\{\omega \in \Omega : \text{ there exists } N_0 = N_0(\omega, \delta) \in \mathbb{N} \text{ such that}$$

$$\sup_{N\geq N_0} \sup_{\substack{N_j\in\mathbb{Z}_{\geq -1}\\ j=1,\cdots,k}} \bigg(\prod_{j=1}^k \langle N_j\rangle^\beta\bigg) |S_{k,N}^{s,b,\delta}(\omega)| \leq \delta^\kappa\bigg\}.$$

Then, we have

$$P(\mathcal{A}_k^c) < e^{-\frac{1}{\delta^c}} \tag{A.14}$$

for any $0 < \delta < \delta_0$.

Proof. In the following, we take $s < 0$ and $\beta > 0$ both sufficiently close to 0 and $b < \frac{1}{2}$ sufficiently close to $\frac{1}{2}$.

We first prove (A.13). Fix $K \gg 1$. Given small $\varepsilon > 0$, it follows from Lemma 2.7 that there exists $\Omega_K \subset \Omega$ with

$$P(\Omega_K^c) \leq e^{-cK^2} \tag{A.15}$$

such that we have

$$|g_n^N(\omega)| \leq K\langle n\rangle^\varepsilon \tag{A.16}$$

for any $\omega \in \Omega_K$, any $n \in \mathbb{Z}$, and any $N \in \mathbb{N}$. We separately consider the following two cases:

$$\text{(i) } n_{\max}^{\frac{1}{2}} \lesssim K \qquad \text{and} \qquad \text{(ii) } n_{\max}^{\frac{1}{2}} \gg K,$$

where $n_{\max}$ is as in (3.11). Suppose that $n_{\max}^{\frac{1}{2}} \lesssim K$. By crudely estimating the contribution in this case with (A.16), $N_1 < |n_1| \lesssim K^2$, and $\widehat{\eta}_\delta(\tau) = \delta\widehat{\eta}(\delta\tau)$, we have

$$\begin{aligned}
\sup_{N\in\mathbb{N}} \sup_{N_1\in\mathbb{Z}_{\geq -1}} \langle N_1\rangle^\beta |S_{1,N}^{s,b,\delta}(\omega)| &\lesssim K^{1-8s+2\beta+2\varepsilon} \bigg\| \mathbf{1}_{n_{\max}\lesssim K} \frac{\widehat{\eta}_\delta(\tau + \Phi(\bar{n}) - |g_{n_1}^N(\omega)|^2)}{\langle\tau\rangle^b} \bigg\|_{\ell^2_{n,n_2,n_3} L^2_\tau} \\
&\lesssim K^{1-8s+2\beta+2\varepsilon} \bigg\| \mathbf{1}_{n_{\max}\lesssim K} \frac{\delta}{\langle\tau\rangle^b \delta^{\frac{1}{2}-b+\varepsilon} \langle\tau + \Phi(\bar{n}) - |g_{n_1}^N(\omega)|^2\rangle^{\frac{1}{2}-b+\varepsilon}} \bigg\|_{\ell^2_{n,n_2,n_3} L^2_\tau} \\
&\lesssim \delta^{\frac{1}{2}+b-\varepsilon} K^{4-8s+2\beta+2\varepsilon} \ll \delta^{\frac{1}{2}+b-\varepsilon} K^5,
\end{aligned} \tag{A.17}$$

provided that $K \gg 1$ and s, β, and ε are all sufficiently close to 0.

Next, we consider the case $n_{\max}^{\frac{1}{2}} \gg K$. In this case, we have

$$\big|\Phi(\bar n) - |g_{n_1}^N(\omega)|^2\big| \sim |\Phi(\bar n)|$$

uniformly for any $\omega \in \Omega_K$, $\bar n = (n_1, n_2, n_3, n) \in \mathbb{Z}^4$, and $N \in \mathbb{N}$. Then, by Lemma 2.4, we have

$$\begin{aligned} \bigg\| \frac{\widehat{\eta}_\delta(\tau + \Phi(\bar n) - |g_{n_1}^N(\omega)|^2)}{\langle \tau \rangle^b} \bigg\|_{L^2_\tau} &\lesssim \bigg(\int \frac{\delta^2}{\langle \tau \rangle^{2b} \delta^{2b} \langle \tau + \Phi(\bar n) - |g_{n_1}^N(\omega)|^2 \rangle^{2b}} d\tau \bigg)^{\frac{1}{2}} \\ &\lesssim \frac{\delta^{1-b}}{\langle \Phi(\bar n) - |g_{n_1}^N(\omega)|^2 \rangle^{2b - \frac{1}{2}}} \sim \frac{\delta^{1-b}}{\langle \Phi(\bar n) \rangle^{2b-\frac{1}{2}}} \end{aligned} \tag{A.18}$$

for $b < \frac{1}{2}$ sufficiently close to $\frac{1}{2}$. Hence, from (A.18) and Lemma 2.1, we have

$$\begin{aligned} &\sup_{N \in \mathbb{N}} \sup_{N_1 \in \mathbb{Z}_{\geq -1}} \langle N_1 \rangle^\beta |S_{1,N}^{s,b,\delta}(\omega)| \\ &\quad \lesssim \delta^{1-b} K \bigg\| \mathbf{1}_{(n_1,n_2,n_3) \in \Gamma(n)} \cdot \mathbf{1}_{|n_1| \geq N_1} \frac{\langle n_1 \rangle^\beta}{n_{\max}^{4b-1+4s-\varepsilon} \langle n - n_1 \rangle^{2b - \frac{1}{2}} \langle n - n_3 \rangle^{2b-\frac{1}{2}}} \bigg\|_{\ell^2_{n,n_2,n_3}} \\ &\quad \lesssim \delta^{1-b} K, \end{aligned} \tag{A.19}$$

provided that s, β, and ε are all sufficiently close to 0 and that $b < \frac{1}{2}$ is sufficiently close to $\frac{1}{2}$. Hence, by choosing $K = \delta^{-\frac{c}{2}}$ for some small $c > 0$, the bound (A.13) follows from (A.15), (A.17), and (A.19).

Let us now turn to the proof of (A.14) for $k = 2$. We have

$$S_{2,N}^{s,b,\delta}(\omega) = \bigg\| \sum_{k_1,k_2=0}^\infty \sum_{\substack{n_1,n_2 \in \mathbb{Z} \\ (n_1,n_2,n_3) \in \Gamma(n)}} \chi_{1,2} \prod_{j=1}^2 \frac{|g_{n_j}^N|^{2k_j} g_{n_j}^*}{k_j!} \frac{\partial^{k_1+k_2} \widehat{\eta}_\delta(\tau + \Phi(\bar n))}{\langle n_3 \rangle^s \langle n \rangle^{2s} \langle \tau \rangle^b} \bigg\|_{\ell^2_{n,n_3} L^2_\tau},$$

where $g_{n_j}^*$ is as in (2.14) and $\chi_{1,2} = \prod_{j=1}^2 \mathbf{1}_{|n_j| > N_j}$. By Minkowski's integral inequality and Lemma 2.11 with (2.3), we have

$$\|S_{2,N}^{s,b,\delta}\|_{L^p(\Omega)} \leq p\delta \sum_{k_1,k_2=0}^\infty (Cp\delta)^{k_1+k_2} \bigg\| \chi_{1,2} \frac{\partial^{k_1+k_2} \widehat{\eta}(\delta(\tau + \Phi(\bar n)))}{\langle n_3 \rangle^s \langle n \rangle^{2s} \langle \tau \rangle^b} \bigg\|_{\ell^2_{n,\Gamma(\bar n)} L^2_\tau}. \tag{A.20}$$

We separately consider the following two cases:

$$\text{(i) } \langle \tau \rangle \gtrsim |\Phi(\bar n)| \qquad \text{and} \qquad \text{(ii) } \langle \tau \rangle \ll |\Phi(\bar n)|.$$

First, suppose that $\langle \tau \rangle \gtrsim |\Phi(\bar n)|$. By Plancherel's identity with (3.21), we have

$$\delta^{\frac{1}{2}} \big\| \partial^k \widehat{\eta}(\delta(\tau + \Phi(\bar n))) \big\|_{L^2_\tau} \leq C^k \tag{A.21}$$

for any $k \in \mathbb{Z}_{\geq 0}$. Then, from (A.20), (A.21), Lemma 2.1, and choosing $p = \delta^{-\theta}$ for some $\theta > 0$ such that $Cp\delta < 1$ as in (3.23), we obtain

$$\begin{aligned} \|S_{2,N}^{s,b,\delta}\|_{L^p(\Omega)} &\leq p\delta^{\frac{1}{2}} \langle N_1 \rangle^{-2\beta} \langle N_2 \rangle^{-2\beta} \sum_{k_1,k_2=0}^\infty (Cp\delta)^{k_1+k_2} \\ &\qquad \times \bigg(\sum_{n \in \mathbb{Z}} \sum_{\Gamma(n)} \frac{\langle n_1 \rangle^{4\beta} \langle n_2 \rangle^{4\beta}}{\langle n_3 \rangle^{2s} \langle n \rangle^{4s} n_{\max}^{4b} (n - n_1)^{2b} (n - n_3)^{2b}} \bigg)^{\frac{1}{2}} \\ &\leq Cp\delta^{\frac{1}{2}} \langle N_1 \rangle^{-2\beta} \langle N_2 \rangle^{-2\beta}, \end{aligned} \tag{A.22}$$

provided that s and β are all sufficiently close to 0 and that $b < \frac{1}{2}$ is sufficiently close to $\frac{1}{2}$.

Next, we consider the case $\langle \tau \rangle \ll |\Phi(\bar{n})|$. By Hausdorff-Young's inequality, we have

$$\big\|\partial^k \widehat{\eta}(\delta(\tau + \Phi(\bar{n})))\big\|_{L^\infty_\tau} \leq \big\|(-it)^k \eta(t)\big\|_{L^1_t} \leq C^k,$$
$$\big\|\delta(\tau + \Phi(\bar{n}))\partial^k \widehat{\eta}(\delta(\tau + \Phi(\bar{n})))\big\|_{L^\infty_\tau} \leq \big\|\partial_t\big((-it)^k \eta(t)\big)\big\|_{L^1_t} \leq C^k$$

for any $k \geq 0$. By interpolating the two estimates above, we have

$$\big\|\delta^{\frac{1}{2}}(\tau + \Phi(\bar{n}))^{\frac{1}{2}} \partial^k \widehat{\eta}(\delta(\tau + \Phi(\bar{n})))\big\|_{L^\infty_\tau} \leq C^k \tag{A.23}$$

for any $k \geq 0$. Then, from (A.23), Lemma 2.1 and choosing $p = \delta^{-\theta}$ as above, we obtain

$$\begin{aligned}
\|S^{s,b,\delta}_{2,N}\|_{L^p(\Omega)} &\leq p\delta^{\frac{1}{2}} \langle N_1\rangle^{-2\beta} \langle N_2\rangle^{-2\beta} \sum_{k_1,k_2=0}^{\infty} (Cp\delta)^{k_1+k_2} \\
&\quad \times \bigg\| \chi_{1,2} \frac{\langle n_1\rangle^{4\beta} \langle n_2\rangle^{4\beta}}{\langle n_3\rangle^s \langle n\rangle^{2s} |\Phi(\bar{n})|^{\frac{1}{2}-\varepsilon} \langle \tau\rangle^{b+\varepsilon}} \bigg\|_{\ell^2_{n,\Gamma(\bar{n})} L^2_\tau} \\
&\leq Cp\delta^{\frac{1}{2}} \langle N_1\rangle^{-2\beta} \langle N_2\rangle^{-2\beta},
\end{aligned} \tag{A.24}$$

provided that s, β, and ε are all sufficiently close to 0 and that $b < \frac{1}{2}$ is sufficiently close to $\frac{1}{2}$ such that $b + \varepsilon > \frac{1}{2}$. By applying Chebyshev's inequality with (A.22) and (A.24) and choosing $\lambda = Cp^2\delta^{\frac{1}{2}}$ with $p = \delta^{-\theta}$, we obtain

$$\begin{aligned}
P\Big(\langle N_1\rangle^\beta \langle N_2\rangle^\beta |S^{s,b,\delta}_{2,N,N_1,N_2}| > \lambda\Big) &\leq \frac{1}{\langle N_1\rangle^{\beta p} \langle N_2\rangle^{\beta p}} C^p \lambda^{-p} p^p \delta^{\frac{p}{2}} \\
&= \frac{1}{\langle N_1\rangle^{\beta p} \langle N_2\rangle^{\beta p}} e^{-p \ln p} = \frac{1}{\langle N_1\rangle^{\beta p} \langle N_2\rangle^{\beta p}} e^{-\frac{1}{\delta^c}}.
\end{aligned} \tag{A.25}$$

Here, we added subscripts N_1 and N_2 in $S^{s,b,\delta}_{2,N,N_1,N_2}$ to show its dependence on N_1 and N_2 explicitly. Now, by summing (A.25) over $N_1, N_2 \in \mathbb{Z}_{\geq -1}$ we obtain

$$P\bigg(\sup_{\substack{N_j \in \mathbb{Z}_{\geq -1} \\ j=1,2}} \langle N_1\rangle^\beta \langle N_2\rangle^\beta |S^{s,b,\delta}_{2,N,N_1,N_2}| > \delta^{\frac{1}{2}-2\theta}\bigg) \leq Ce^{-\frac{1}{\delta^c}}$$

for any $0 < \delta < \delta_0$, where $\delta_0 > 0$ is defined by $\beta\delta_0^{-\theta} = 1$.

Let $M \geq N \geq 1$. Then, by slightly modifying the computation above with the definition (4.8) of g^N_n and Minkowski's inequality (on the $\ell^2_{n,n_3} L^2_\tau$-norm), we also have

$$\|S^{s,b,\delta}_{2,M,N_1,N_2} - S^{s,b,\delta}_{2,N,N_1,N_2}\|_{L^p(\Omega)} \leq Cp\delta^{\frac{1}{2}} N^{-\beta} \langle N_1\rangle^{-2\beta} \langle N_2\rangle^{-2\beta},$$

since we must have $n_{\max} \geq N$ to have a non-zero contribution to the left-hand side above. This shows that $\{S^{s,b,\delta}_{2,N,N_1,N_2}\}_{N\in\mathbb{N}}$ forms a Cauchy sequence in $L^p(\Omega)$ for any $p \geq 1$ and converges to some limit $S^{s,b,\delta}_{2,\infty,N_1,N_2}$, satisfying

$$\|S^{s,b,\delta}_{2,\infty,N_1,N_2} - S^{s,b,\delta}_{2,N,N_1,N_2}\|_{L^p(\Omega)} \leq Cp\delta^{\frac{1}{2}} N^{-\beta} \langle N_1\rangle^{-2\beta} \langle N_2\rangle^{-2\beta}$$

and

$$P\bigg(\sup_{\substack{N_j \in \mathbb{Z}_{\geq -1} \\ j=1,2}} \langle N_1\rangle^\beta \langle N_2\rangle^\beta |S^{s,b,\delta}_{2,\infty,N_1,N_2}| > \delta^{\frac{1}{2}-2\theta}\bigg) \leq Ce^{-\frac{1}{\delta^c}} \tag{A.26}$$

for any $0 < \delta < \delta_0$.

By repeating the computation in (A.25), we then obtain

$$P\bigg(\sup_{\substack{N_j\in\mathbb{Z}_{\geq 0}\\ j=1,2}}\langle N_1\rangle^\beta\langle N_2\rangle^\beta|S^{s,b,\delta}_{2,\infty,N_1,N_2}-S^{s,b,\delta}_{2,N,N_1,N_2}|>\delta^{\frac{1}{2}-2\theta}\bigg)\leq\frac{C}{N^{\beta p}}e^{-\frac{1}{\delta^c}} \tag{A.27}$$

for any $0<\delta<\delta_0$ (by possibly making δ_0 smaller but independent of $N\in\mathbb{N}$). Given $\ell\in\mathbb{N}$ sufficiently large, by choosing $\ell=\delta^{2\theta-\frac{1}{2}}$, it follows from (A.27) that

$$\begin{aligned}\sum_{N=1}^\infty P\bigg(&\sup_{\substack{N_j\in\mathbb{Z}_{\geq 0}\\ j=1,2}}\langle N_1\rangle^\beta\langle N_2\rangle^\beta|S^{s,b,\delta}_{2,\infty,N_1,N_2}-S^{s,b,\delta}_{2,N,N_1,N_2}|>\frac{1}{\ell}\bigg)\\ &\leq\sum_{N=1}^\infty\frac{C(\ell)}{N^{\beta p}}<\infty,\end{aligned}$$

since $\beta p>1$. Therefore, we conclude from the Borel-Cantelli lemma that there exists Ω_ℓ with $P(\Omega_\ell)=1$ such that for each $\omega\in\Omega_\ell$, there exists $N_0=N_0(\omega)\in\mathbb{N}$ such that

$$\sup_{\substack{N_j\in\mathbb{Z}_{\geq 0}\\ j=1,2}}\langle N_1\rangle^\beta\langle N_2\rangle^\beta|S^{s,b,\delta}_{2,\infty,N_1,N_2}-S^{s,b,\delta}_{2,N,N_1,N_2}|\leq\frac{1}{\ell}$$

for any $N\geq N_0$. By setting $\Sigma=\bigcap_{\ell=1}^\infty\Omega_\ell$, we have $P(\Sigma)=1$. This shows that, as $N\to\infty$, $S^{s,b,\delta}_{2,N,N_1,N_2}$ converges almost surely to $S^{s,b,\delta}_{2,\infty,N_1,N_2}$ with respect to the metric:

$$d(f_{N_1,N_2},g_{N_1,N_2}):=\sup_{\substack{N_j\in\mathbb{Z}_{\geq -1}\\ j=1,2}}\langle N_1\rangle^\beta\langle N_2\rangle^\beta|f_{N_1,N_2}-g_{N_1,N_2}|.$$

Combining this almost sure convergence with (A.26), we obtain (A.14) when $k=2$.

The proof of (A.14) for $k=3$ follows in an analogous manner and hence we omit details. □

A.3. **Proof of Lemma 2.11.** We conclude this appendix by presenting the proof of Lemma 2.11.

First, we consider the case $|\mathcal{A}|=1$. By Stirling's formula: $k!\sim\sqrt{k}\big(\frac{k}{e}\big)^k$, there exist $C_0,C>0$ such that

$$\frac{(2k+1)!}{(k!)^2}\leq C_0^k\sqrt{k}\leq C^k \tag{A.28}$$

for any $k\in\mathbb{Z}_{\geq 0}$. Hence, the desired estimate (2.15) follows from the Wiener chaos estimate (Lemma 2.8), (A.1), and (A.28).

The proof when $|\mathcal{A}|\geq 2$ follows in a similar manner, using an estimate such as (A.28). In the following, we only present the proof when $|\mathcal{A}|=3$, namely, $\mathcal{A}=\{1,2,3\}$, since the proof for the case $|\mathcal{A}|=2$ follows in an analogous manner. In this case, by the Wiener chaos estimate (Lemma 2.8) with (2.12), we have

$$\|\Sigma_n\|_{L^p(\Omega)}\leq(p-1)^{k+\frac{3}{2}}\|\Sigma_n\|_{L^2(\Omega)}. \tag{A.29}$$

In the following, we estimate $\|\Sigma_n\|_{L^2(\Omega)}$. From (2.13), we have

$$\|\Sigma_n\|_{L^2(\Omega)} = \frac{1}{k_1!k_2!k_3!}\Bigg\| \sum_{(n_1,n_2,n_3)\in\Gamma(n)} \sum_{(\widetilde{n}_1,\widetilde{n}_2,\widetilde{n}_3)\in\Gamma(n)} c^{\bar{k}}_{n_1,n_2,n_3} \overline{c^{\bar{k}}_{\widetilde{n}_1,\widetilde{n}_2,\widetilde{n}_3}} \\ \times \prod_{j=1}^{3} |g_{n_j}|^{2k_j} g^*_{n_j} \prod_{\widetilde{j}=1}^{3} \overline{|g_{\widetilde{n}_{\widetilde{j}}}|^{2k_j} g^*_{\widetilde{n}_{\widetilde{j}}}} \Bigg\|_{L^2(\Omega)}. \tag{A.30}$$

Recall from (A.1) that under the conditions $n_2 \neq n_1, n_3$ and $\widetilde{n}_2 \neq \widetilde{n}_1, \widetilde{n}_3$, the right-hand side of (A.30) yields zero contribution unless $n_2 = \widetilde{n}_2$. Hence, we assume $n_2 = \widetilde{n}_2$ in the following.

• **Case 1:** $n_1 \neq n_3$. Note that we must have $n_1 = \widetilde{n}_1 \neq \widetilde{n}_3$ or $n_1 = \widetilde{n}_3 \neq \widetilde{n}_1$ in this case. Otherwise, the right-hand side of (A.30) yields zero contribution.

We first consider the case $n_1 = \widetilde{n}_1 \neq \widetilde{n}_3$. In this case, we have $n_3 = \widetilde{n}_3$. Then, from (A.1), we obtain

$$\begin{aligned} \text{RHS of (A.30)} &\leq \frac{1}{k_1!k_2!k_3!}\Bigg(\sum_{\Gamma(n)} |c^{\bar{k}}_{n_1,n_2,n_3}|^2 \prod_{j=1}^{3}(2k_j+1)!\Bigg)^{\frac{1}{2}} \\ &\leq C^k \Bigg(\sum_{\Gamma(n)} |c^{\bar{k}}_{n_1,n_2,n_3}|^2\Bigg)^{\frac{1}{2}}. \end{aligned} \tag{A.31}$$

Next, we consider the case $n_1 = \widetilde{n}_3 \neq \widetilde{n}_1$. In this case, we have $n_3 = \widetilde{n}_1$. Then, from (A.1) and (A.28), we obtain

$$\text{RHS of (A.30)} \leq \frac{1}{k_1!k_2!k_3!}\Bigg(\sum_{\Gamma(n)} |c^{\bar{k}}_{n_1,n_2,n_3}|^2 [(k_1+k_3+1)!]^2 (2k_2+1)!\Bigg)^{\frac{1}{2}}. \tag{A.32}$$

We claim that

$$\frac{(k_1+k_3+1)!}{k_1!k_3!} \leq C^{k_1+k_3} \tag{A.33}$$

for some $C > 0$. Hence, from (A.32) with (A.28) and (A.33), we obtain

$$\text{RHS of (A.30)} \leq C^k \Bigg(\sum_{\Gamma(n)} |c^{\bar{k}}_{n_1,n_2,n_3}|^2\Bigg)^{\frac{1}{2}}. \tag{A.34}$$

Hence, it remains to prove (A.32). Without loss of generality, assume $k_1 \leq k_3$. Then, by Stirling's formula, we have

$$\frac{(k_1+k_3+1)!}{k_1!k_3!} \leq C^{k_3} \frac{(k_1+k_3)^{\frac{3}{2}}}{\sqrt{k_1k_3}} \frac{(k_1+k_3)^{k_1}}{k_1^{k_1}} \leq C^{k_1+k_3}\Bigg[\bigg(1+\frac{k_3}{k_1}\bigg)^{\frac{k_1}{k_3}}\Bigg]^{k_3}. \tag{A.35}$$

Then, (A.33) follows from (A.35) once we note that $\lim_{x\to\infty}(1+x)^{\frac{1}{x}} = 1$.

• **Case 2:** $n_1 = n_3$. In this case, we must have $n_1 = n_3 = \widetilde{n}_1 = \widetilde{n}_3$. Proceeding as before with (A.1), we have

$$\begin{aligned}\text{RHS of (A.30)} &\le \frac{1}{k_1!k_2!k_3!}\bigg(\sum_{\Gamma(n)} |c^{\bar{k}}_{n_1,n_2,n_3}|^2 (2k_1+2k_3+2)!(2k_2+1)!\bigg)^{\frac{1}{2}}\\ &\le C^k \bigg(\sum_{\Gamma(n)} |c^{\bar{k}}_{n_1,n_2,n_3}|^2\bigg)^{\frac{1}{2}}, \end{aligned} \tag{A.36}$$

where we used

$$\frac{(2k_1+2k_3+2)!}{(k_1!)^2(k_3!)^2} \le C^{k_1+k_3} \tag{A.37}$$

in the second inequality. The proof of (A.37) is analogous to that of (A.33) and thus we omit details.

Putting (A.29), (A.31), (A.34), and (A.36) together, we obtain (2.15) when $\mathcal{A} = \{1,2,3\}$. This completes the proof of Lemma 2.11.

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